## Lecture 11: Application: The Mach-Zehnder interferometer

- Coherent-state input
- Squeezed-state input

Mach-Zehnder interferometer with coherent-state input: Now we apply our knowledge about quantum-state transformations to interferometry where we are interested in phase differences. A common example that is also known from classical optics is the MachZehnder interferometer in which two beam are split at one beam splitter and later recombined at another beam splitter (see Fig. 16). Any phase difference that the two beams might have picked up between the two beam splitters will appear as an interference pattern.


FIG. 16: A Mach-Zehnder interferometer consists of two symmetric beam splitters BS1 and BS2 and two detectors D1 and D2. Light in one arm of the interferometer experiences a phase shift $\varphi$.

In our first example, let us consider the case where the two inputs of the first beam splitter are prepared in a coherent state $|\alpha\rangle$ and the vacuum state $|0\rangle$, respectively. Hence,

$$
\begin{equation*}
\left|\psi_{\text {in }}\right\rangle=|0, \alpha\rangle . \tag{11.1}
\end{equation*}
$$

Assuming a symmetric beam splitter, its transmission matrix is

$$
\mathbf{T}=\frac{1}{\sqrt{2}}\left(\begin{array}{cc}
1 & 1  \tag{11.2}\\
-1 & 1
\end{array}\right)
$$

After the first beam splitter, the quantum state has transformed into

$$
\begin{equation*}
\left|\psi_{\text {in }}\right\rangle=|0, \alpha\rangle \mapsto\left|\frac{\alpha}{\sqrt{2}}, \frac{\alpha}{\sqrt{2}}\right\rangle . \tag{11.3}
\end{equation*}
$$

The phase shift in the lower arm will cause the coherent amplitude to be replaced by $\alpha_{2} \mapsto$ $e^{i \varphi} \alpha_{2}$. Hence, the quantum state transforms into

$$
\begin{equation*}
\left|\frac{\alpha}{\sqrt{2}}, \frac{\alpha}{\sqrt{2}}\right\rangle \mapsto\left|\frac{\alpha}{\sqrt{2}}, \frac{e^{i \varphi} \alpha}{\sqrt{2}}\right\rangle . \tag{11.4}
\end{equation*}
$$

Finally, the second beam splitter reverses the transformation (11.3) and produces a state

$$
\begin{equation*}
\left|\psi_{\text {out }}\right\rangle=\left|\frac{\alpha}{2}\left(e^{i \varphi}+1\right), \frac{\alpha}{2}\left(e^{i \varphi}-1\right)\right\rangle . \tag{11.5}
\end{equation*}
$$

These three steps are pictorially represented in Fig. 17 as a series of phase-space snapshots.


FIG. 17: The input state is a product of a coherent states (blue) and the vacuum state (red). The effect of the beam splitters and the phase shifter is to displace the coherent states in phase space.

In the Mach-Zehnder interferometer, one is interested in the difference of the two photocurrents at the outputs. We have seen before in connection with homodyne tomography that this current gives information about the phase difference between the two interferometer arms. From Eq. (11.5) we find that

$$
\begin{equation*}
\langle\hat{O}\rangle=\left\langle\hat{n}_{1}\right\rangle-\left\langle\hat{n}_{2}\right\rangle=|\alpha|^{2} \cos \varphi . \tag{11.6}
\end{equation*}
$$

Moreover, the phase error we make by measuring the difference current can be estimated by the error propagation formula

$$
\begin{equation*}
\Delta \varphi=\frac{\langle\Delta \hat{O}\rangle}{\left|\frac{\partial\langle\hat{O}\rangle}{\partial \varphi}\right|} . \tag{11.7}
\end{equation*}
$$

For the Mach-Zehnder interferometer we find that

$$
\begin{equation*}
\Delta \varphi=\frac{1}{|\alpha||\sin \varphi|} \tag{11.8}
\end{equation*}
$$

and therefore the minimum value is $\Delta \varphi_{\text {min }}=1 /|\alpha|$ for $\varphi=\pi / 2$ which is the working point of the interferometer. Because $|\alpha|=\sqrt{\langle\hat{n}\rangle}$, the phase error decreases with the intensity of the coherent input field as in the classical interferometer.
Mach-Zehnder interferometer with squeezed-state input: Let us now assume that one of the input ports of the Mach-Zehnder interferometer was not prepared in the vacuum state, but in the squeezed vacuum state $|\xi\rangle$. Although we could use the input-output
relations as before, the resulting quantum state would look rather complicated. However, because we are dealing with Gaussian states, their respective Wigner functions must be Gaussian functions as well and are thus easier to handle.

Recall that the Wigner function for a single-mode coherent state $\left|\alpha_{0}\right\rangle$ and for the squeezedvacuum state can be written as [recall Eqs. (7.5) and (7.7)]

$$
\begin{equation*}
W_{\mathrm{in}}\left(\alpha_{1}, \alpha_{2}\right)=\frac{4}{\pi^{2}} e^{-2\left|\alpha_{1}-\alpha_{0}\right|^{2}} \exp \left[-2\left(\frac{\alpha_{2}^{\prime 2}}{e^{-2 \xi}}+\frac{\alpha_{2}^{\prime \prime 2}}{e^{2 \xi}}\right)\right] . \tag{11.9}
\end{equation*}
$$

Let us now define the vector of quadratures as $\boldsymbol{\lambda}=\left(\alpha_{1}^{\prime}, \alpha_{1}^{\prime \prime}, \alpha_{2}^{\prime}, \alpha_{2}^{\prime \prime}\right)^{T}$ and the displacement vector as $\mathbf{d}=\left(\alpha_{0}^{\prime}, \alpha_{0}^{\prime \prime}, 0,0\right)^{T}$. Then, the Wigner function (11.9) can be written in the compact form

$$
\begin{equation*}
W_{\text {in }}\left(\alpha_{1}, \alpha_{2}\right)=\frac{4}{\pi^{2}} e^{-2\left|\alpha_{0}\right|^{2}} e^{-2 \boldsymbol{\lambda}^{T} \cdot \boldsymbol{V} \cdot \boldsymbol{\lambda}+2 \mathrm{~d}^{T} \cdot \boldsymbol{\lambda}+2 \boldsymbol{\lambda}^{T} \cdot \mathrm{~d}} \tag{11.10}
\end{equation*}
$$

where

$$
\boldsymbol{V}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{11.11}\\
0 & 1 & 0 & 0 \\
0 & 0 & e^{2 \xi} & 0 \\
0 & 0 & 0 & e^{-2 \xi}
\end{array}\right)
$$

is the (inverse of the) covariance matrix. Since a beam splitter, by Eq. (9.14), acts linearly on the quadrature components, we can write it as a matrix transformation $\boldsymbol{\lambda} \mapsto \boldsymbol{B} \cdot \boldsymbol{\lambda}$. Similarly, the phase shifter acts as $\boldsymbol{\lambda} \mapsto \boldsymbol{\Phi} \cdot \boldsymbol{\lambda}$ where both matrices are given by

$$
\boldsymbol{B}=\frac{1}{\sqrt{2}}\left(\begin{array}{cccc}
1 & 0 & -1 & 0  \tag{11.12}\\
0 & 1 & 0 & -1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{array}\right) \quad \text { and } \quad \boldsymbol{\Phi}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cos \varphi & -\sin \varphi \\
0 & 0 & \sin \varphi & \cos \varphi
\end{array}\right)
$$

The quadrature components then transform under the action of the Mach-Zehnder interferometer into $\boldsymbol{\lambda} \mapsto \boldsymbol{\lambda}^{\prime}=(\boldsymbol{B} \cdot \boldsymbol{\Phi} \cdot \boldsymbol{B}) \cdot \boldsymbol{\lambda}$ which in turn means that the covariance matrix $\boldsymbol{V}$ has changed into

$$
\begin{equation*}
\boldsymbol{V} \mapsto \boldsymbol{V}^{\prime}=(\boldsymbol{B} \cdot \boldsymbol{\Phi} \cdot \boldsymbol{B})^{T} \cdot \boldsymbol{V} \cdot(\boldsymbol{B} \cdot \boldsymbol{\Phi} \cdot \boldsymbol{B}) \tag{11.13}
\end{equation*}
$$

and the displacement vector into $\mathbf{d} \mapsto \mathrm{d}^{\prime}=(\boldsymbol{B} \cdot \boldsymbol{\Phi} \cdot \boldsymbol{B}) \cdot \mathbf{d}$. Inserted into the Wigner function (11.9) we obtain the Wigner function of the quantum state after passing through the interferometer as

$$
\begin{equation*}
W_{\text {out }}\left(\alpha_{1}, \alpha_{2}\right)=\frac{4}{\pi^{2}} e^{-2\left|\alpha_{0}\right|^{2}} e^{-2 \boldsymbol{\lambda}^{T} \cdot \boldsymbol{V}^{\prime} \cdot \boldsymbol{\lambda}+2 \mathbf{d}^{T} \cdot \boldsymbol{\lambda}+2 \boldsymbol{\lambda}^{T} \cdot \mathbf{d}^{\prime}} . \tag{11.14}
\end{equation*}
$$

Recall that the Wigner function can be used to compute the expectation value of any (symmetrically ordered) operator [Eq. (7.8)], for example,

$$
\begin{equation*}
\left\langle\hat{a}_{1}^{\dagger m} \hat{a}_{1}^{n}\right\rangle_{\text {sym }}=\int d^{2} \alpha_{1} d^{2} \alpha_{2} \alpha_{1}^{* m} \alpha_{1}^{n} W\left(\alpha_{1}, \alpha_{2}\right) . \tag{11.15}
\end{equation*}
$$

The powers $\alpha_{1}^{* m} \alpha_{1}^{n}$ can be obtained by multiplying the Wigner function with a factor $e^{\left(\mathbf{j}^{T} \cdot \boldsymbol{\lambda}+\boldsymbol{\lambda}^{T} \cdot \mathbf{j}\right) / 2}$, differentiating with respect to the elements $j_{i}$ an appropriate number of times and setting $\mathbf{j}=0$ at the end of the calculation. For example, the expectation value of the number operator can be obtained as

$$
\begin{align*}
\left\langle\hat{a}_{1}^{\dagger} \hat{a}_{1}\right\rangle_{\mathrm{sym}} & =\int d^{2} \alpha_{1} d^{2} \alpha_{2}\left|\alpha_{1}\right|^{2} W\left(\alpha_{1}, \alpha_{2}\right)=\int d^{2} \alpha_{1} d^{2} \alpha_{2}\left(\alpha_{1}^{\prime 2}+\alpha_{1}^{\prime \prime 2}\right) W\left(\alpha_{1}, \alpha_{2}\right) \\
& =\left.\left(\frac{\partial^{2}}{\partial j_{1}^{2}}+\frac{\partial^{2}}{\partial j_{2}^{2}}\right) \int d^{2} \alpha_{1} d^{2} \alpha_{2} W\left(\alpha_{1}, \alpha_{2}\right) e^{\left(\mathbf{j}^{T} \cdot \boldsymbol{\lambda}+\boldsymbol{\lambda}^{T} \cdot \mathbf{j}\right) / 2}\right|_{\mathbf{j}=0} \\
& =\left.\left(\frac{\partial^{2}}{\partial j_{1}^{2}}+\frac{\partial^{2}}{\partial j_{2}^{2}}\right) e^{\left(4 \mathbf{d}^{T}+\mathbf{j}^{T}\right) \cdot \boldsymbol{V}^{\prime-1} \cdot(\mathbf{d}+\mathbf{j}) / 8-2\left|\alpha_{0}\right|^{2}}\right|_{\mathbf{j}=0} . \tag{11.16}
\end{align*}
$$

In the last step, we have performed the Gaussian integration.
What remains is to compute the difference current $\left\langle\hat{n}_{1}\right\rangle-\left\langle\hat{n}_{2}\right\rangle$ and the phase error estimate. To do so, we must not forget that the Wigner function can only be used to compute expectation values of symmetrically ordered operators. Thus, we have to convert all operators into their respective symmetrically ordered form by using the commutation relation $\left[\hat{a}, \hat{a}^{\dagger}\right]=1$ [see Eq. (2.12)]. We find that

$$
\begin{equation*}
\hat{n}=\left\{\hat{a}^{\dagger} \hat{a}\right\}_{\mathrm{sym}}-\frac{1}{2}, \quad \hat{n}^{2}=\left\{\hat{a}^{\dagger 2} \hat{a}^{2}\right\}_{\mathrm{sym}}-\left\{\hat{a}^{\dagger} \hat{a}\right\}_{\mathrm{sym}} . \tag{11.17}
\end{equation*}
$$

The differentiation with respect to $j_{i}$ is best done using some computer algebra program. The result is that, at the working point $\varphi=\pi / 2$, the phase error is

$$
\begin{equation*}
\Delta \varphi_{\min }=\frac{\left(\sinh ^{2} \xi+\alpha_{0}^{\prime 2} e^{2 \xi}+\alpha_{0}^{\prime \prime 2} e^{-2 \xi}\right)^{1 / 2}}{\left|\sinh ^{2} \xi-\left|\alpha_{0}\right|^{2}\right|} . \tag{11.18}
\end{equation*}
$$

If we assume that $\alpha_{0}^{\prime}=0$ and $\alpha_{0}^{\prime \prime} \gg e^{4 \xi}$ which is usually the case, then the phase error is approximately

$$
\begin{equation*}
\Delta \varphi_{\min } \simeq \frac{e^{-\xi}}{\left|\alpha_{0}^{\prime \prime}\right|} \tag{11.19}
\end{equation*}
$$

Compared with the coherent-state input, the phase resolution increases by a factor of $e^{\xi}$ which makes squeezed states a highly valuable resource in interferometry. A particular example includes gravitational-wave detection [J. Gea-Banacloche and G. Leuchs, J. Mod. Opt. 34, 793 (1987)].

## Lecture 13: Conditional linear-optical networks

- Principles of quantum-state engineering
- Designing unitary operations

Principles of quantum-state engineering: Beam splitters and phase shifters are used to apply unitary transformations to a given quantum state. This immediately restricts the class of final states one can reach, given that the class of initial states is in general limited (if it wasn't, then there would be no need for elaborate transformations). Typically, available resources consist of coherent states produced by lasers, squeezed states produced by parametric processes in some nonlinear crystals, and single-photon Fock states from atomic emission processes. But what if, for some reason, one needs a state that is a coherent superposition of photon-number states, such as

$$
\begin{equation*}
|\psi\rangle=c_{0}|0\rangle+c_{1}|1\rangle+c_{2}|2\rangle+\ldots, \tag{13.1}
\end{equation*}
$$

given the finite resources mentioned above? For a general set of coefficient $c_{i}$, the state (13.1) cannot be produced by unitary transformations of the above-mentioned set of states.

The additional, non-unitary, element that has to be added to achieve the desired transformation is a projective measurement such as a photon counting measurement. Let us assume that we are given a coherent state $|\alpha\rangle$ and a single-photon Fock state $|1\rangle$. We overlap both at a beam splitter with (complex) transmittivity $T$ and reflectivity $R$ and project one output onto the vacuum state (Fig. 19).


FIG. 19: Principal building block for adding a photon to a quantum state, here $|\alpha\rangle$. The beam splitter is highly transmitting, $|T| \approx 1$.


FIG. 20: Simplest way of building the nonlinear sign shift gate with three beam splitters and two additional light modes.

By projection we mean that we measure the photon number in a photon counter and, by finding no photon, regard the transformation as successful. However, there are potentially many possible measurement outcomes out of which one has to be selected. Hence, there is always a probability of less than one (usually rather small) associated with the correct measurement result.

The initial two-mode state is $\left|\psi_{12}\right\rangle=\left|\alpha_{1}, 1_{2}\right\rangle=\hat{D}_{1}(\alpha) \hat{a}_{2}^{\dagger}\left|0_{1}, 0_{2}\right\rangle$. Application of the inputoutput relations (9.6) then yields

$$
\begin{equation*}
\left|\psi_{12}\right\rangle \mapsto\left(R \hat{a}_{1}^{\dagger}+T^{*} \hat{a}_{2}^{\dagger}\right) e^{\alpha\left(T \hat{a}_{1}^{\dagger}-R^{*} \hat{a}_{2}^{\dagger}\right)-\alpha\left(T^{*} \hat{a}_{1}-R \hat{a}_{2}\right)}\left|0_{1}, 0_{2}\right\rangle \tag{13.2}
\end{equation*}
$$

Projected onto the vacuum state $\left\langle 0_{2}\right|$ yields the (un-normalized) state

$$
\begin{equation*}
\left|\tilde{\psi}_{1}\right\rangle \propto R \hat{a}_{1}^{\dagger} \hat{D}_{1}(T \alpha)\left|0_{1}\right\rangle . \tag{13.3}
\end{equation*}
$$

The probability of obtaining this state, i.e. the probability of finding no photon in the photon counter, is

$$
\begin{equation*}
p=\left\langle\tilde{\psi}_{1} \mid \tilde{\psi}_{1}\right\rangle . \tag{13.4}
\end{equation*}
$$

If we assume the beam splitter transmittivity to be very large, $|T| \rightarrow 1$, then the state (13.3) is the photon-added state $\left|\tilde{\psi}_{1}\right\rangle \propto \hat{a}_{1}^{\dagger}\left|\alpha_{1}\right\rangle$. The success probability $p$, however, because it is proportional to $|R|^{2}=1-|T|^{2}$, will be very small.

But we see from this example that with conditional state preparation we can effectively generate non-unitary transformations (such as $\hat{a}^{\dagger}$ ) acting on arbitrary quantum states. This simple scheme can be generalized to generate arbitrary finite superpositions of Fock states. Suppose the expansion of the state (13.1) terminates at the $N$ th Fock state. Then it can be written as

$$
\begin{equation*}
|\psi\rangle=\left(\hat{a}^{\dagger}-\alpha_{N}^{*}\right)\left(\hat{a}^{\dagger}-\alpha_{N-1}^{*}\right) \cdots\left(\hat{a}^{\dagger}-\alpha_{2}^{*}\right)\left(\hat{a}^{\dagger}-\alpha_{1}^{*}\right)|0\rangle \tag{13.5}
\end{equation*}
$$

where the complex numbers $\alpha_{n}^{*}$ are the roots of the characteristic polynomial

$$
\sum_{n=0}^{N} \frac{c_{n}}{\sqrt{n!}}\left(\alpha^{*}\right)^{n}=0 .
$$

Recalling how the displacement operator transforms the photonic amplitude operators, $\hat{D}(\alpha) \hat{a}^{\dagger} \hat{D}^{\dagger}=\hat{a}^{\dagger}-\alpha^{*}$, the state can be written as a succession of displacement and creation operators acting on the vacuum state,

$$
\begin{equation*}
|\psi\rangle=\hat{D}\left(\alpha_{N}\right) \hat{a}^{\dagger} \hat{D}^{\dagger}\left(\alpha_{N}\right) \hat{D}\left(\alpha_{N-1}\right) \hat{a}^{\dagger} \hat{D}^{\dagger}\left(\alpha_{N-1}\right) \cdots \hat{D}\left(\alpha_{1}\right) \hat{a}^{\dagger} \hat{D}^{\dagger}\left(\alpha_{1}\right)|0\rangle . \tag{13.6}
\end{equation*}
$$

We already know how to generate creation operators by conditional measurements. Displacing a state in phase space is even simpler, one merely has to overlap it with a strong coherent state at a weakly reflecting beam splitter. With those two ingredients, we have all the tools in hand necessary to generate arbitrary finite superpositions of photon-number states [M. Dakna, J. Clausen, L. Knöll, and D.-G. Welsch, Phys. Rev. A 59, 1658 (1999)]. Designing unitary operations: Although beam splitters generate unitary operations, they cannot generate any conceivable unitary operation. For example, it is impossible to find a beam splitter of a set of beam splitters and phase shifters that can provide the transformation

$$
\begin{equation*}
c_{0}|0\rangle+c_{1}|1\rangle+c_{2}|2\rangle \mapsto c_{0}|0\rangle+c_{1}|1\rangle-c_{2}|2\rangle . \tag{13.7}
\end{equation*}
$$

This transformation is indeed unitary as it preserves the norm of the state. Since it is a single-mode state, the only optical element that realizes unitary operations is a phase shifter which can only generate transformations to states $c_{0}|0\rangle+e^{i \varphi} c_{1}|1\rangle+e^{2 i \varphi} c_{2}|2\rangle$ which does not contain the transformation (13.7).

To solve this problem, we can use similar tricks as in quantum-state engineering. However, the engineering is at a higher level in that we are not content with producing a particular quantum state, but we aim to generate a unitary transformation that it exactly the same for a whole class of quantum states specified by the coefficients $\left\{c_{0}, c_{1}, c_{2}\right\}$. For this purpose, let us look at the beam splitter network depicted in Fig. 20. It consists of three beam splitters whose reflectivities are assumed to be real (we use the phase convention that reflection from the top of BS2 and from the bottom of BS1 and BS3 induced a sign change). The transformation is said to be successful if the two detectors find a single photon and the vacuum state, respectively.

In principle, we could now use our knowledge of general quantum-state transformations to find the transformed output state and determine the beam splitter transmittivities from that. There is, however, a simpler method by adding up all the probability amplitudes of all paths leading to one photon in the upper detector and no photon in the lower detector [T.C. Ralph, A.G. White, W.J. Munro, and G.J. Milburn, Phys. Rev. A 65, 012314 (2001)]. Suppose no photon enters the network from the state $|\psi\rangle$. Then there are two ways how the auxiliary photon can end up in the upper detector, by reflecting off all three beam splitters or by passing through BS1 and BS3. The total probability amplitude for this to happen is
therefore

$$
\begin{equation*}
\sqrt{p}=R_{1} R_{2} R_{3}+T_{1} T_{3} . \tag{13.8}
\end{equation*}
$$

If one photon comes in from $|\psi\rangle$, either this photon or the auxiliary photon must leave the network through the open output port. The probability amplitude for the sum all these possibilities must be the same as (13.8) because it should be state-independent. Hence, it must be

$$
\begin{equation*}
\sqrt{p}=R_{1} R_{3} T_{2}^{2}-R_{2}\left(R_{1} R_{2} R_{3}+T_{1} T_{3}\right)=R_{1} R_{3} T_{2}^{2}-\sqrt{p} R_{2} \tag{13.9}
\end{equation*}
$$

where we used Eq. (13.8). Therefore, we find that

$$
\begin{equation*}
\sqrt{p}=\frac{R_{1} R_{3} T_{2}^{2}}{1+R_{2}} . \tag{13.10}
\end{equation*}
$$

Finally, if two photon arrive at the network from $|\psi\rangle$, two photons from either $|\psi\rangle$ and/or the auxiliary photon must leave through the open output port. The probability amplitude is now $-\sqrt{p}$ because we want to realize the transformation (13.7) which reads

$$
\begin{equation*}
-\sqrt{p}=-2 R_{1} R_{2} R_{3} T_{2}^{2}+R_{2}^{2} \sqrt{p} \tag{13.11}
\end{equation*}
$$

Combining Eqs. (13.10) and (13.11) yields a condition for the reflectivity of BS2 as

$$
\begin{equation*}
R_{2}^{2}=(\sqrt{2}-1)^{2} \text {. } \tag{13.12}
\end{equation*}
$$

Maximizing the success probability $p$ over the other two beam splitter angles yields $p_{\max }=1 / 4$ for $R_{1}=R_{3}=(4-2 \sqrt{2})^{-1}$. That is, only in one quarter of all cases will we find the correct measurement outcome and hence will we have implemented the correct transformation (13.7). But note again that the success probability is state-independent! This simple example shows how, in principle, arbitrary unitary transformations can be implemented by conditional measurement scheme using auxiliary photons, beam splitters and photon counters. Such schemes are extensively investigated in connection with all-optical quantum information processing and computing.

