Lecture 9: The lossless beam splitter

- input-output relations
- quantum-state transformation at beam splitters

Input-output relations: So far, we have characterized important classes of quantum states in terms of their eigenvalues and eigenvectors, as well as in terms of their photon statistics. In the following lectures, we will see how one can manipulate quantum states of light with linear optical elements. In particular, we will concentrate on non-absorbing beam splitters. If we neglect the three-dimensional character of the electromagnetic fields and focus on one-dimensional propagation only, we can regard a beam splitter simply as a dielectric plate, possibly consisting of several different layers (see Figure 11).

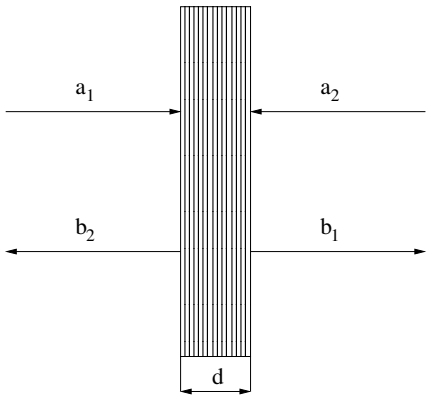

FIG. 11: Simple model of a beam splitter with thickness d, possibly consisting of several layers of different dielectric materials.

For our purpose, it is enough to consider only a single dielectric plate of thickness d and restrict ourselves to one particular polarization and propagation along the x-direction. In this case, the electric-field strength turns into a scalar operator $\hat{E}(x) = i\int dk\, c|k|A(k,x)\hat{a}(k)$ $-i\int dk\, c|k|A^*(k,x)\hat{a}^\dagger(k)$. The Helmholtz equation (1.18) for the (scalar) mode functions $A(k,x)$ is then an ordinary differential equation

$$\frac{\partial^2}{\partial x^2}A(k,x) + n^2(x)k^2A(k,x) = 0\,, \tag{9.1}$$

where the refractive index $n(x)$ is n for $-\frac{d}{2} \leq x \leq \frac{d}{2}$ and 1 for $|x| > \frac{d}{2}$. The solutions of this equation are again plane waves that can be constructed similar to the quantum-mechanical

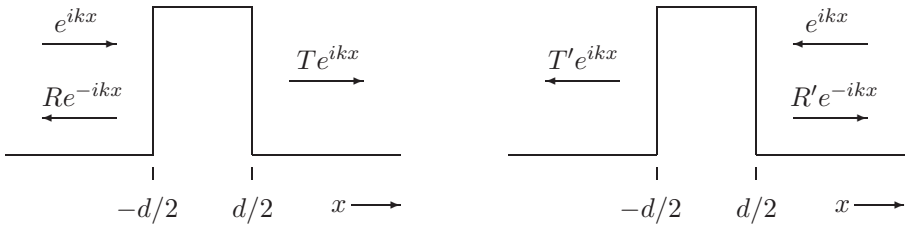

FIG. 12: A plane wave e^{ikx} with $k > 0$ (left figure) or $k < 0$ (right figure) impinges onto a beam splitter from the left or right, respectively, and splits into transmitted and reflected parts.

wave functions at a potential barrier. If an incoming plane wave e^{ikx} from the left ($k > 0$) hits the barrier, it will split into a reflected part $R(\omega)e^{-ikx}$ and a transmitted part $T(\omega)e^{ikx}$ (see Fig. 12). Similar things happen if a plane wave e^{ikx} impinges onto the dielectric plate from the right ($k < 0$). The reflection coefficient $R(\omega)$ and the transmission coefficient $T(\omega)$ depend on the details on the beam splitter. Hence, the mode functions can be written as

$$A(k,x) = \sqrt{\frac{\hbar}{4\pi\varepsilon_0\omega A}} \begin{cases} e^{ikx} + R(\omega)e^{-ikx} & x \leq -\frac{d}{2} \\ T(\omega)e^{ikx} & x \geq \frac{d}{2} \end{cases} \qquad k > 0\,, \tag{9.2}$$

$$A(k,x) = \sqrt{\frac{\hbar}{4\pi\varepsilon_0\omega A}} \begin{cases} T'(\omega)e^{ikx} & x \leq -\frac{d}{2} \\ e^{ikx} + R'(\omega)e^{-ikx} & x \geq \frac{d}{2} \end{cases} \qquad k < 0\,, \tag{9.3}$$

where, in principle, the transmission and reflection coefficients for light impinging onto the beam splitter from the light or right could be different.

We see from Eqs. (9.2) and (9.3) that the mode functions $A(k,x)$ are decomposed into incoming and outgoing plane waves. When inserting these mode function back into the definition of the electric-field strength operator, we might just as well decompose the field itself into incoming and outgoing waves. If we denote the incoming fields by

$$\hat{E}_{\text{in}}(x) = i\int_0^\infty d\omega \sqrt{\frac{\hbar\omega}{4\pi\varepsilon_0 cA}} \left[e^{ix\frac{\omega}{c}}\hat{a}_1(\omega) + e^{-ix\frac{\omega}{c}}\hat{a}_2(\omega)\right] + \text{h.c.} \tag{9.4}$$

and the outgoing field by

$$\hat{E}_{\text{out}}(x) = i\int_0^\infty d\omega \sqrt{\frac{\hbar\omega}{4\pi\varepsilon_0 cA}} \left[e^{ix\frac{\omega}{c}}\hat{b}_1(\omega) + e^{-ix\frac{\omega}{c}}\hat{b}_2(\omega)\right] + \text{h.c.}\,, \tag{9.5}$$

then, by inspection of Eqs. (9.2) and (9.3), we can relate the photonic amplitude operators

of incoming and outgoing waves as

$$\boxed{\begin{aligned} \hat{b}_1(\omega) &= T(\omega)\hat{a}_1(\omega) + R'(\omega)\hat{a}_2(\omega) \\ \hat{b}_2(\omega) &= R(\omega)\hat{a}_1(\omega) + T'(\omega)\hat{a}_2(\omega) \end{aligned}} . \tag{9.6}$$

We can write the input-output relations (9.6) also in matrix form. Defining the vectors $\hat{\mathbf{a}}(\omega) = [\hat{a}_1(\omega), \hat{a}_2(\omega)]^T$ and $\hat{\mathbf{b}}(\omega) = [\hat{b}_1(\omega), \hat{b}_2(\omega)]^T$, Eq. (9.6) turns into

$$\hat{\mathbf{b}}(\omega) = \boldsymbol{T}(\omega) \cdot \hat{\mathbf{a}}(\omega) , \tag{9.7}$$

where the transmission matrix $\boldsymbol{T}(\omega)$ is defined by

$$\boldsymbol{T}(\omega) = \begin{pmatrix} T(\omega) & R'(\omega) \\ R(\omega) & T'(\omega) \end{pmatrix} . \tag{9.8}$$

The transmission and reflection coefficients from both sides are, however, not independent from one another. Recall that the photonic amplitude operators satisfy bosonic commutation relations. If this is true for the amplitude operators $\hat{a}_i(\omega)$ and $\hat{a}_i^\dagger(\omega)$ of the incoming fields, the same must hold for those of the outgoing fields, $\hat{b}_i(\omega)$ and $\hat{b}_i^\dagger(\omega)$. The commutation relations impose constraints on the range of values that the transmission and reflection coeffients can take. In particular, we find that

$$\left[\hat{b}_1(\omega), \hat{b}_1^\dagger(\omega)\right] := 1 = |T(\omega)|^2 + |R'(\omega)|^2 , \tag{9.9}$$

$$\left[\hat{b}_2(\omega), \hat{b}_2^\dagger(\omega)\right] := 1 = |T'(\omega)|^2 + |R(\omega)|^2 , \tag{9.10}$$

$$\left[\hat{b}_2(\omega), \hat{b}_1^\dagger(\omega)\right] := 0 = T^*(\omega)R(\omega) + R'^*(\omega)T'(\omega) . \tag{9.11}$$

From these relations we see that both magnitudes and the phases of the coefficients are constrained. Equations (9.9) and (9.10) imply that the total probability of transmitting or reflecting a photon is always unity. This expresses photon-number conservation (or energy conservation) at a lossless beam splitter. The phase relation (9.11) implies that $|T| = |T'|$ and $|R| = |R'|$. Finally, a solution to Eq. (9.11) is obtained if we set $R' = -R^*$ and $T' = T^*$, so that the transmission matrix takes the final form

$$\boldsymbol{T}(\omega) = \begin{pmatrix} T(\omega) & R(\omega) \\ -R^*(\omega) & T^*(\omega) \end{pmatrix} . \tag{9.12}$$

Note that the transmission matrix is unitary which reflects energy conservation, and that its determinant is $\det \boldsymbol{T}(\omega) = 1$. Mathematically, this means that $\boldsymbol{T}(\omega) \in \mathrm{SU}(2)$.

Quantum-state transformation at beam splitters: We have seen so far how the photonic amplitude operators transform at a beam splitter. The resulting input-output relations (9.6) can be used to derive a simple rule how to transfrom quantum states. We have noted previously in connection with the Wigner function, that the density operator $\hat{\varrho}$ can be regarded as an operator function of the photonic amplitude operators $\hat{a}_i(\omega)$ and $\hat{a}_i^\dagger(\omega)$, $\hat{\varrho}_{\text{in}} \equiv \hat{\varrho}_{\text{out}}[\hat{\mathbf{a}}(\omega), \hat{\mathbf{a}}^\dagger(\omega)]$. If we are now given such a density operator that depends functionally on the amplitude operators of the incoming fields, $\hat{\mathbf{a}}(\omega)$ and $\hat{\mathbf{a}}^\dagger(\omega)$, then we have to replace those with the amplitude operators of the outgoing fields, $\hat{\mathbf{b}}(\omega)$ and $\hat{\mathbf{b}}^\dagger(\omega)$, to obtain the transformed density operator. This means that the transformed density operator reads

$$\boxed{\hat{\varrho}_{\text{out}}\left[\hat{\mathbf{a}}(\omega), \hat{\mathbf{a}}^\dagger(\omega)\right] = \hat{\varrho}_{\text{in}}\left[\boldsymbol{T}^+(\omega)\cdot\hat{\mathbf{a}}(\omega), \boldsymbol{T}^T(\omega)\cdot\hat{\mathbf{a}}^\dagger(\omega)\right]}, \tag{9.13}$$

where we have used the inverse transformation to Eq. (9.6). Hence, the transformation of the quantum state (in the form of the density operator), Eq. (9.13), is the inverse of the transformation of the photonic amplitude operators (9.6).

Transformation of the Wigner function: Finally, we will see how the Wigner function transforms at a beam splitter. Recall that it is defined for single-mode fields as $W(\alpha) = \text{Tr}[\hat{\varrho}\hat{\delta}(\hat{a}-\alpha)]$. Because we need two (spatial) modes to describe the effect of a beam splitter, the definition of the Wigner function has to be extended to cover this situation. This is done by defining a two-dimensional vector $\boldsymbol{\alpha} = [\alpha_1, \alpha_2]^T$ and writing $W(\boldsymbol{\alpha}) = \text{Tr}[\hat{\varrho}\hat{\delta}(\hat{\mathbf{a}}-\boldsymbol{\alpha})]$.

If we denote the Wigner function of the incoming two-mode quantum state by $W_{\text{in}}(\boldsymbol{\alpha})$ and the Wigner function of the outgoing quantum state by $W_{\text{out}}(\boldsymbol{\alpha})$, they are related by ($\omega$-dependence suppressed for notational convenience)

$$\begin{aligned}
W_{\text{out}}(\boldsymbol{\alpha}) &= \text{Tr}\left\{\hat{\varrho}_{\text{out}}\left[\hat{\mathbf{a}}, \hat{\mathbf{a}}^\dagger\right]\hat{\delta}(\hat{\mathbf{a}}-\boldsymbol{\alpha})\right\} \\
&\overset{(9.6)}{=} \text{Tr}\left\{\hat{\varrho}_{\text{in}}\left[\boldsymbol{T}^+\cdot\hat{\mathbf{a}}, \boldsymbol{T}^T\cdot\hat{\mathbf{a}}^\dagger\right]\hat{\delta}(\hat{\mathbf{a}}-\boldsymbol{\alpha})\right\} \\
&\overset{\hat{\mathbf{a}}\mapsto\boldsymbol{T}\cdot\hat{\mathbf{a}}}{=} \text{Tr}\left\{\hat{\varrho}_{\text{in}}\left[\hat{\mathbf{a}}, \hat{\mathbf{a}}^\dagger\right]\hat{\delta}(\mathbf{T}\cdot\hat{\mathbf{a}}-\boldsymbol{\alpha})\right\} \\
&\overset{\boldsymbol{\alpha}\mapsto\boldsymbol{T}^+\cdot\boldsymbol{\alpha}}{=} \text{Tr}\left\{\hat{\varrho}_{\text{in}}\left[\hat{\mathbf{a}}, \hat{\mathbf{a}}^\dagger\right]\hat{\delta}(\hat{\mathbf{a}}-\mathbf{T}^+\cdot\boldsymbol{\alpha})\right\} \\
&= W_{\text{in}}(\mathbf{T}^+\cdot\boldsymbol{\alpha})\,.
\end{aligned} \tag{9.14}$$

Hence, the Wigner function transforms in the same way as the density operator, namely with the inverse of the transformation matrix $\mathbf{T}(\omega)$. Intuitively, this must be so because the Wigner function and the density operator contain the same information and are transformable into one another.

Lecture 10: Examples of quantum-state transformations

- Coherent states, homodyne detection
- Number states, Hong–Ou–Mandel quantum interference

Quantum-state transformation of coherent states: The first and simplest example of a quantum-state transformation is obtained by looking at the situation where the incoming radiation field is prepared in a two-mode coherent state $|\boldsymbol{\alpha}\rangle \equiv |\alpha_1, \alpha_2\rangle$. The density operator of the two-mode coherent state can then be written as

$$\hat{\varrho}_{\rm in} = |\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}| = \hat{D}(\boldsymbol{\alpha})|0,0\rangle\langle 0,0|\hat{D}^\dagger(\boldsymbol{\alpha})\,, \tag{10.1}$$

where $\hat{D}(\boldsymbol{\alpha}) \equiv \hat{D}(\alpha_1)\hat{D}(\alpha_2)$ is the two-mode displacement operator which, by inspection of Eq. (5.1), is an operator function of the amplitude operators $\hat{a}$ and $\hat{a}^\dagger$. If we write $\hat{D}(\boldsymbol{\alpha})$ as

$$\hat{D}(\boldsymbol{\alpha}) = e^{\boldsymbol{\alpha}^T\cdot\hat{\mathbf{a}}^\dagger - \boldsymbol{\alpha}^+\cdot\hat{\mathbf{a}}}\,, \tag{10.2}$$

we can apply the quantum-state transformation formula (9.13) to it and replace $\hat{\mathbf{a}}$ by $\boldsymbol{T}^+\cdot\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}^\dagger$ by $\boldsymbol{T}^T\cdot\hat{\mathbf{a}}^\dagger$. This is equivalent to replacing $\boldsymbol{\alpha}$ by $\boldsymbol{T}\cdot\boldsymbol{\alpha}$ since

$$e^{\boldsymbol{\alpha}^T\cdot\boldsymbol{T}^T\cdot\hat{\mathbf{a}}^\dagger - \boldsymbol{\alpha}^+\cdot\boldsymbol{T}^+\cdot\hat{\mathbf{a}}} = e^{(\boldsymbol{T}\cdot\boldsymbol{\alpha})^T\cdot\hat{\mathbf{a}}^\dagger - (\boldsymbol{T}\cdot\boldsymbol{\alpha})^+\cdot\hat{\mathbf{a}}}\,. \tag{10.3}$$

This transforms the displacement operator into $\hat{D}(\boldsymbol{T}\cdot\boldsymbol{\alpha})$ and we obtain

$$\hat{\varrho}_{\rm out} = |\boldsymbol{T}\cdot\boldsymbol{\alpha}\rangle\langle\boldsymbol{T}\cdot\boldsymbol{\alpha}| = |T\alpha_1 + R\alpha_2, -R^*\alpha_1 + T^*\alpha_2\rangle\langle T\alpha_1 + R\alpha_2, -R^*\alpha_1 + T^*\alpha_2|\,, \tag{10.4}$$

which says that a two-mode coherent state transforms into another two-mode coherent state with their coherent amplitudes changed according to the input-output relations (9.6),

$$\begin{aligned}\alpha_1' &= T\alpha_1 + R\alpha_2\,,\\ \alpha_2' &= -R^*\alpha_1 + T^*\alpha_2\,.\end{aligned}$$

This relation shows yet again why the coherent states can be regarded as those quantum states that have classical counterparts because classical light would behave exactly as in Eq. (10.4). The classical amplitudes would independently transform according to the rules set by classical optics, and so do the coherent amplitudes of a coherent state. Other quantum states, however, will behave completely differently as we will see later.

Balanced homodyne detection: Let us now return to the beam splitter transformation (9.6) and see what happens to the photon number at the output ports of the beam splitter. From (9.6) we easily calculate

$$\begin{aligned}
\hat{b}_1^\dagger \hat{b}_1 &= (T^* \hat{a}_1^\dagger + R^* \hat{a}_2^\dagger)(T\hat{a}_1 + R\hat{a}_2) &&= |T|^2 \hat{a}_1^\dagger \hat{a}_1 + |R|^2 \hat{a}_2^\dagger \hat{a}_2 + T^* R \hat{a}_1^\dagger \hat{a}_2 + T R^* \hat{a}_2^\dagger \hat{a}_1 \,, \\
\hat{b}_2^\dagger \hat{b}_2 &= (-R^* \hat{a}_1^\dagger + T^* \hat{a}_2^\dagger)(-R\hat{a}_1 + T\hat{a}_2) &&= |R|^2 \hat{a}_1^\dagger \hat{a}_1 + |T|^2 \hat{a}_2^\dagger \hat{a}_2 - T^* R \hat{a}_1^\dagger \hat{a}_2 - T R^* \hat{a}_2^\dagger \hat{a}_1 \,.
\end{aligned} \tag{10.5}$$

Let us now assume that we have a $50\% : 50\%$ beam splitter with $T = |T|e^{i\varphi_T}$, $R = |R|e^{i\varphi_R}$ and $|T| = |R| = 1/\sqrt{2}$. Moreover, let the radiation field in input mode 2, the so-called local oscillator, be prepared in a coherent state $|\alpha_L\rangle$ with $\alpha_L = |\alpha_L|e^{i\varphi_L}$. Then, the difference between the expectation values of the photon numbers in the output ports of the beam splitter is

$$\langle \hat{b}_1^\dagger \hat{b}_1 \rangle - \langle \hat{b}_2^\dagger \hat{b}_2 \rangle = |\alpha_L| \langle \hat{a}_1 e^{i(\varphi_T - \varphi_R - \varphi_L)} + \hat{a}_1^\dagger e^{-i(\varphi_T - \varphi_R - \varphi_L)} \rangle = |\alpha_L| \langle \hat{x}(\varphi) \rangle \tag{10.6}$$

with $\varphi = \varphi_T - \varphi_R - \varphi_L$. Since the current in a photodiode is proportional to the number of photons impinging on it, the lhs of Eq. (10.6) represents the difference of the two photocurrents at the detectors D1 and D2 (see Fig. 13).

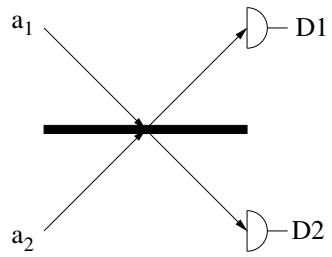

FIG. 13: Scheme of a balanced homodyne detection set up. A local oscillator in a coherent state $|\alpha_L\rangle$ and an unknown quantum state impinge on a $50\% : 50$ beam splitter.

The photocurrent difference gives then, via the rhs of Eq. (10.6), information about the phase-rotated quadrature of the signal mode 1. The phase φ can be adjusted by tuning the phase φ_L of the local oscillator. In this way, we obtain information about all phase-rotated quadratures of the signal state. This is the principle behind *balanced homodyne detection* which is used to reconstruct quantum states of the radiation field. [Note that we only derived that the mean values of the photocurrents give information about the expectation value of

the quadrature component. However, it can be shown that for a strong local oscillator the complete quadrature statistics $p(x, \varphi)$ is measured from which the Wigner function and hence the density operator can be reconstructed.]

Quantum-state transformation of number states: We will now turn to the transformation of quantum states that do not have any classical counterparts, namely the number states. Let us assume that the radiation field was initially prepared in a product state of two single-photon Fock states, $|\psi_{\rm in}\rangle = |1,1\rangle$. On using the transmission matrix (9.12), we obtain for the transformed state

$$\begin{aligned}
|\psi_{\rm out}\rangle &= \hat{a}_1^\dagger \hat{a}_2^\dagger |0,0\rangle \\
&= (T\hat{b}_1^\dagger - R^*\hat{b}_2^\dagger)(R\hat{b}_1^\dagger + T^*\hat{b}_2^\dagger)|0,0\rangle \\
&= \left[TR(\hat{b}_1^\dagger)^2 - T^*R^*(\hat{b}_2^\dagger)^2 + \left(|T|^2 - |R|^2\right)\hat{b}_1^\dagger\hat{b}_2^\dagger\right]|0,0\rangle \\
&= \sqrt{2}TR|2,0\rangle - \sqrt{2}T^*R^*|0,2\rangle + \left(|T|^2 - |R|^2\right)|1,1\rangle\,. \qquad (10.7)
\end{aligned}$$

In contrast to the previous example, the quantum state (10.7) is not a product state anymore. In particular, for a symmetric beam splitter, the $|1,1\rangle$-contribution we had initially started with, vanishes completely. This is a result of *quantum interference* in which probability amplitudes and not the probabilities add themselves up. In case of a symmetric beam splitter, we can visualise the possible paths that the two photons can take (see Fig. 14). The two photons, here labelled in green and red colours (despite their indistinguishability),

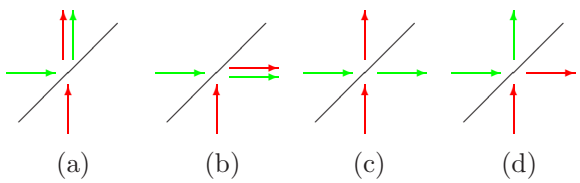

FIG. 14: Different pathways two indistinguishable photons impinging on a beam splitter can take.

can end up both at the same detector [cases (a) and (b)] or at different detectors [cases (c) and (d)]. However, in case (d) a phase shift of π occurs due to reflection from an interface, and hence the amplitudes of (c) and (d) cancel each other. This is the origin of the Hong-Ou-Mandel quantum interference effect.

Imagine the incoming photons were classical particles (or waves) which would be subject to the same transmission and reflection coefficients as in Eq. (9.12). Then each of the

classical particles would be transmitted through the beam splitter with a probability $|T|^2$ and reflected from it with a probability $|R|^2$. We can collect the results for classical and quantum particles in the following table:

output port 1	output port 2	quantum prob.	classical prob.
2	0	$2\|T\|^2\|R\|^2$	$\|T\|^2\|R\|^2$
0	2	$2\|T\|^2\|R\|^2$	$\|T\|^2\|R\|^2$
1	1	$(\|T\|^2 - \|R\|^2)^2$	$\|T\|^4 + \|R\|^4$

TABLE I: Probabilities of finding a given number of particles in both output ports of a beam splitter. Note that both columns associated with quantum and classical particles add up to unity as they should.

From Eq. (10.7) and Table I we see that for a symmetric beam splitter the output state is an equal superposition of the states $|2,0\rangle$ and $|0,2\rangle$,

$$|\psi_{\text{out}}\rangle = \frac{1}{\sqrt{2}}\left(|2,0\rangle - |0,2\rangle\right) \tag{10.8}$$

[assuming $T, R \in \mathbb{R}$]. The suppression of the $|1,1\rangle$-component in a coincidence measurement at both outputs is also named *Hong–Ou–Mandel* dip after the authors who first verified this effect [C.K. Hong, Z.Y. Ou, and L. Mandel, Phys. Rev. Lett. **59**, 2044 (1987)].

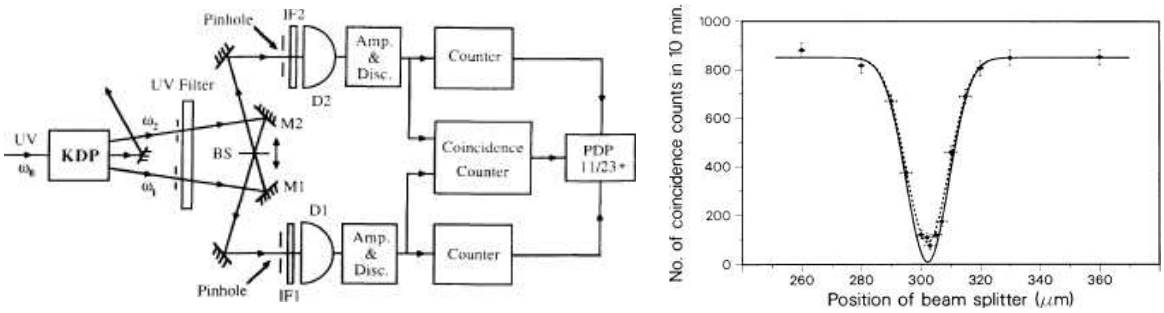

FIG. 15: In the original experiment, a bi-photon was created by parametric down-conversion in a KDP crystal, and the temporal overlap changed by displacing the beam splitter BS. When the two photons overlap perfectly, the coincidence counts go to zero. Displacing the beam splitter shifts the spatiotemporal mode profiles of the two photons with respect to each other. The two photons thus do not overlap perfectly and the coincidence counts increase.

Lecture 12: Principle of quantum-state reconstruction

- phase-rotated quadrature distribution
- reconstruction of the Wigner function

Phase-rotated quadrature distribution: In a previous lecture, we have seen that, with the help of a symmetric beam splitter and a strong local oscillator, we can measure the expectation value of the phase-rotated quadrature $\hat{x}(\varphi)$. Such a procedure is known as balanced homodyne detection; a signal light field is mixed at a symmetric beam splitter with a local oscillator prepared in a strong coherent state $|\alpha_L\rangle$ with $\alpha_L = |\alpha_L|e^{i\varphi_L}$. The difference in the expectation values of the photon counts in both detectors (see Fig. 13) is proportional to the expectation value of $\hat{x}(\varphi = \varphi_T - \varphi_R - \varphi_L)$ [see Eq. 10.6],

$$\langle \hat{n}_1 \rangle - \langle \hat{n}_2 \rangle = |\alpha_L| \langle \hat{x}(\varphi) \rangle \, .$$

The notion of expectation value (or mean value) implies that, only after many measurements on identically prepared quantum states, we will obtain $\langle \hat{x}(\varphi) \rangle$ *on average*. Any single photodetection will result in a measurement outcome that is very probably not exactly $\langle \hat{x}(\varphi) \rangle$. Repeated measurements will yield a probability distribution $p(x, \varphi)$ with mean value $\langle \hat{x}(\varphi) \rangle$, spread $\langle \hat{x}^2(\varphi) \rangle$ and so forth. Knowledge of all moments implies knowledge of the full probability distribution and vice versa.

Each of these moments $\langle \hat{x}^n(\varphi) \rangle$ can then be calculated from the phase-rotated probability distribution as

$$\langle \hat{x}^n(\varphi) \rangle = \int_{-\infty}^{\infty} dx \, x^n p(x, \varphi) \, . \tag{12.1}$$

More generally, these moments are generated by an exponential function of the form

$$\langle e^{iz\hat{x}(\varphi)} \rangle = \int_{-\infty}^{\infty} dx \, e^{izx} p(x, \varphi) \tag{12.2}$$

by differentiating with respect to z at $z = 0$,

$$\langle \hat{x}^n(\varphi) \rangle = (-i)^n \frac{d^n}{dz^n} \langle e^{iz\hat{x}(\varphi)} \rangle_{z=0} \, . \tag{12.3}$$

Equation (12.2) is just a Fourier transform and can be inverted to give

$$p(x, \varphi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz \, e^{-izx} \langle e^{iz\hat{x}(\varphi)} \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz \, e^{-izx} \langle \hat{D}\left(ize^{-i\varphi}\right) \rangle \tag{12.4}$$

where, in the last equation, we have used the definitions of the quadrature operator [Eq. (5.15)] and the displacement operator [Eq. (5.1)].

We have noted back in Lecture 6 that the displacement operator is the Fourier transformed operator-valued δ function. Hence, we can further write

$$p(x,\varphi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dz \int d^2\alpha\, e^{-izx} e^{iz\alpha e^{i\varphi} + iz\alpha^* e^{-i\varphi}} W(\alpha)\,. \tag{12.5}$$

The integral over z can now be performed, and after a change of variables and denoting $W(\alpha) \equiv W(\alpha', \alpha'')$ we obtain the important relation

$$p(x,\varphi) = \int_{-\infty}^{\infty} dy\, W(x\cos\varphi + y\sin\varphi, -x\sin\varphi + y\cos\varphi)\,. \tag{12.6}$$

Equation (12.6) implies that there is a one-to-one relation between the (measurable) phase-rotated probability distribution and the Wigner function of the quantum state. It also says that $p(x,\varphi)$ is a marginal probability distribution which is obtained by integrating the Wigner function along a certain direction in phase space specified by the angle φ. For example, for $\varphi = 0$, Eq. (12.6) simplifies to $p(x,0) = \int dy\, W(x,y)$, which projects the Wigner function onto the x-axis.

Reconstruction of the Wigner function: In the same way as we found that the quadrature probability distribution can be obtained from the Wigner function, the Wigner function can be retrieved from the quadrature distribution. To see this, let us go back to Eq. (12.2) and write it as

$$\langle e^{iz\hat{x}(\varphi)}\rangle = \mathrm{Tr}\left[\hat{\varrho}\hat{D}\left(ize^{-i\varphi}\right)\right] = \int_{-\infty}^{\infty} dx\, e^{izx} p(x,\varphi)\,. \tag{12.7}$$

The displacement operator is the Fourier transform of the operator-valued δ function, hence

$$\mathrm{Tr}\left[\hat{\varrho}\hat{D}\left(ize^{-i\varphi}\right)\right] = \int d^2\alpha\, e^{\alpha^*\gamma - \alpha\gamma^*} W(\alpha)$$

where $\gamma = ize^{-i\varphi}$. Fourier transforming again leaves us with

$$W(\beta) = \frac{1}{\pi^2}\int d^2\gamma\, e^{\gamma^*\beta - \gamma\beta^*} \int_{-\infty}^{\infty} dx\, e^{izx} p(x,\varphi)\,. \tag{12.8}$$

Transforming the phase-space integration over γ to polar co-ordinates $d^2\gamma \mapsto zdzd\varphi$, and using the symmetry relation $p(x,\varphi) = p(-x, \pi-\varphi)$ which follows from inspection of Eq. (12.6),

finally obtain

$$W(\beta', \beta'') = \frac{1}{4\pi^2} \int_0^{\pi} d\varphi \iint_{-\infty}^{\infty} dx\, dz\, p(x, \varphi) e^{iz(x - \beta' \cos\varphi + \beta'' \sin\varphi)} |z| \,. \tag{12.9}$$

Equation (12.9) is now our main result. Together with its inverse relation, Eq. (12.6), it shows that we can reconstruct the Wigner function of a quantum state, and therefore its density operator, from the phase-rotated quadrature probability distribution $p(x, \varphi)$. If the quadrature distribution is known for all angles in the interval $[0, \pi]$, then the Wigner function is also completely known.

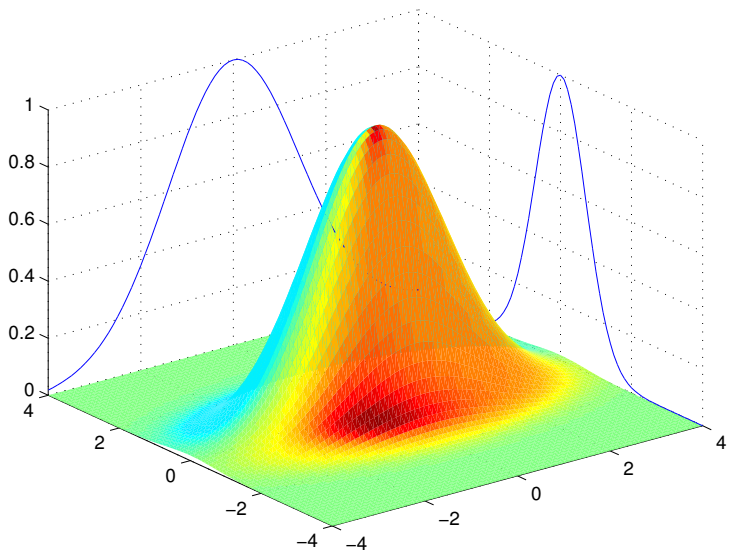

FIG. 18: Wigner function of a squeezed state with marginal distributions (phase-rotated quadrature distributions for $\varphi = 0$ and $\varphi = \pi/2$).