Lecture 3: Casimir effect

- Casimir force between parallel plates — heuristic arguments
- Exact calculation

Casimir force between parallel plates — heuristic arguments: Earlier, we have argued that the infinite energy contained in the quantized free electromagnetic field does not have any consequence other than the relocation of the origin of the energy scale. However, in confined geometries with a mode expansion different from that in free space, there will be an effect known as the Casimir effect. The simplest situation in which the Casimir effect can be studied is between two parallel plates with lateral size L that are separated by a distance $d \ll L$ (see Figure 2). In order to calculate the vacuum energy between these plates, we first have to establish the mode structure of the electromagnetic field.

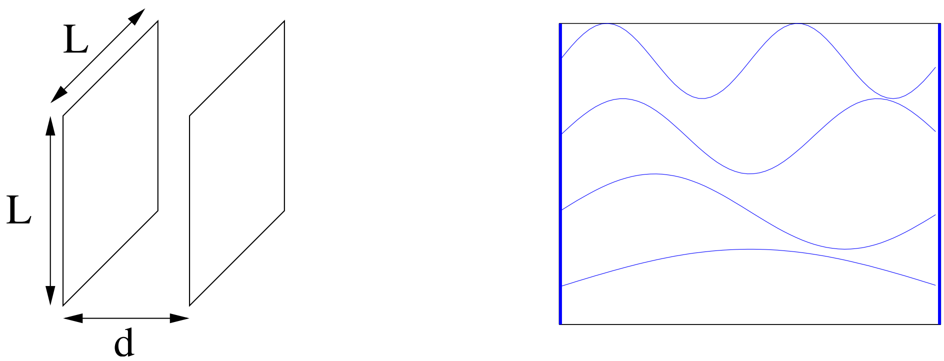

FIG. 2: Two perfectly conducting plates of lateral dimension L are separated by a small distance $d \ll L$. The boundary conditions imply a restriction of modes as shown in the figure on the right.

Assuming that the plates are perfectly conducting, they impose boundary conditions in such a way that the mode functions vanish on the surface of the plates. In particular, the wave vector perpendicular to the plate surface can take only discrete values $k_z = n\pi/d$ ($n = 0, 1, 2, \ldots$). This means that the number of modes that are confined in the space between the plates depends on their separation.

Imagine now a situation in which the plates are separated by a distance d, and let us call the ground-state energy of this configuration $E(d)$. Now we separate the plates by a further amount δd. The corresponding ground-state energy $E(d + \delta d)$ has now increased because more modes have been made available that can contribute to the energy. But this means that the minimal energy configuration is the one in which both plates have vanishing separation, i.e. *there exists an attractive vacuum force.*

We can even estimate the strength of that force by estimating the ground-state energy in the space between the plates in the following way. The ground-state energy must be proportional to the enclosed volume ($L^2 d$), and the summation (or integration) over all allowed wave numbers effectively reaches down to values on the order of $1/d$. Thus,

$$E(d) \propto (L^2 d) \int\limits_{1/d}^{\Lambda} k^2 dk [\hbar\omega(k)] = \hbar c L^2 d \int\limits_{1/d}^{\Lambda} k^3 dk \propto \hbar c L^2 \left(d\Lambda^4 - \frac{1}{d^3} \right) . \tag{3.1}$$

Here, the integral has been cut off at an upper wave number Λ to make the result finite, and we have used that $\omega = kc$. Hence, the force per unit area on the plates, $\mathcal{F} = F/L^2$, must be

$$\mathcal{F} \propto -\frac{\hbar c}{d^4} . \tag{3.2}$$

This is an attractive force that falls off very quickly as d^{-4} and is therefore only relevant at very small distances. For example, for $d = 1\mu$m we find that $\mathcal{F} \propto -0.03\,\mathrm{Nm}^{-2}$, which is correct upto a numerical factor that we will compute next.

Exact calculation: Of course, our heuristic arguments do not give the exact expression for the Casimir force. Neither does it deal with the divergence as the cut-off wave number becomes larger. We therefore need a more elaborate calculation.

The total zero-point energy is

$$E(d) = \sum_\alpha \frac{1}{2}\hbar\omega_\alpha = \frac{\hbar c}{2} \sum_\alpha |\mathbf{k}_\alpha| = \frac{\hbar c}{2} \int L^2 \frac{d^2 k_\parallel}{(2\pi)^2} \left[|\mathbf{k}_\parallel| + 2 \sum_{n=1}^{\infty} \left(\mathbf{k}_\parallel^2 + \frac{n^2\pi^2}{d^2} \right)^{1/2} \right] . \tag{3.3}$$

Here we have used the fact that for each wave vector there are two transverse polarization modes, except for $k_z = 0$ in which case there is only one independent mode. Note that we have written the summation over the transverse modes in integral form which is allowed if the size of the plates can be made arbitrarily large.

The zero-point energy computed in Eq. (3.3) is, of course, infinite. In order to make sense out of it, we subtract the contribution of the free quantized field modes in the same volume,

$$E_0 = \frac{\hbar c}{2} \int L^2 \frac{d^2 k_\parallel}{(2\pi)^2} \int\limits_{-\infty}^{\infty} d\frac{dk_z}{2\pi} 2\sqrt{\mathbf{k}_\parallel^2 + k_z^2} = \frac{\hbar c}{2} \int L^2 \frac{d^2 k_\parallel}{(2\pi)^2} \int\limits_{0}^{\infty} dn\, 2\sqrt{\mathbf{k}_\parallel^2 + \frac{n^2\pi^2}{d^2}} . \tag{3.4}$$

Needless to say, the energy in Eq. (3.4) is also infinite. Introducing polar co-ordinates for the transverse wave vector, we find that the energy difference per unit surface area of the plates is

$$\mathcal{E} = \frac{E(d) - E_0}{L^2} = \frac{\hbar c}{2\pi} \int_0^\infty k\, dk \left(\frac{k}{2} + \sum_{n=1}^{\infty} \sqrt{k^2 + \frac{n^2\pi^2}{d^2}} - \int_0^\infty dn \sqrt{k^2 + \frac{n^2\pi^2}{d^2}} \right) . \tag{3.5}$$

This integral still seems to diverge for large values of k. The trick is now to effectively cut off the integral above a certain value $k > k_{\max}$ and remove the regularization only at the end of the calculation. Physically, this can be justified by noting that for very high frequencies (much larger than the plasma frequency of the plate material) the plates are effectively transparent. We will therefore introduce a cut-off function $f(k)$ with the following properties:

$$f(k) = \begin{cases} 1\,, & k < k_{\max} \\ 0\,, & k \gg k_{\max} \end{cases} . \tag{3.6}$$

With the change of variable to $u = d^2k^2/\pi^2$, we obtain

$$\begin{aligned} \mathcal{E} = \hbar c \frac{\pi^2}{4d^3} \int_0^\infty du \Bigg[& \frac{\sqrt{u}}{2} f\left(\frac{\pi}{d}\sqrt{u}\right) + \sum_{n=1}^{\infty} \sqrt{u+n^2} f\left(\frac{\pi}{d}\sqrt{u+n^2}\right) \\ & - \int_0^\infty dn \sqrt{u+n^2} f\left(\frac{\pi}{d}\sqrt{u+n^2}\right) \Bigg] . \end{aligned} \tag{3.7}$$

Now we introduce a function

$$F(n) = \int_0^\infty du \sqrt{u+n^2} f\left(\frac{\pi}{d}\sqrt{u+n^2}\right) \tag{3.8}$$

with which we can write Eq. (3.7) as

$$\mathcal{E} = \hbar c \frac{\pi^2}{4d^3} \left[\frac{1}{2}F(0) + \sum_{n=1}^{\infty} F(n) - \int_0^\infty dn\, F(n) \right] . \tag{3.9}$$

The expression in brackets can be computed using the Euler–MacLaurin resummation formula which states that

$$\frac{1}{2}F(0) + \sum_{n=1}^{\infty} F(n) - \int_0^\infty dn\, F(n) = -\frac{1}{2!}B_2 F'(0) - \frac{1}{4!}B_4 F'''(0) + \ldots \tag{3.10}$$

where B_n are the Bernoulli numbers with $B_2 = 1/6$, $B_4 = -1/30$. Re-writing the function $F(n)$ as

$$F(n) = \int_{n^2}^\infty du \sqrt{u} f\left(\frac{\pi}{d}\sqrt{u}\right) , \quad F'(n) = -2n^2 f\left(\frac{n\pi}{d}\right) , \tag{3.11}$$

we see that $F'''(0) = -4$, and all other derivatives vanish because of the assumption that the derivatives of f vanish at the origin.

Finally, insering this result into Eq. (3.7) gives

$$\mathcal{E} = \frac{\hbar c \pi^2}{d^3} \frac{B_4}{4!} = -\frac{\pi^2}{720} \frac{\hbar c}{d^3} \,. \tag{3.12}$$

Note that the cut-off function does not appear in this final result. This means that our calculation is in fact independent of it. The force per unit area acting on the plates is then

$$\boxed{\mathcal{F} = -\frac{\partial \mathcal{E}}{\partial d} = -\frac{\pi^2}{240} \frac{\hbar c}{d^4}} \,. \tag{3.13}$$

Equation (3.13) is now the exact result. Inserting the distance $d = 1\,\mu$m as above, we now find that $\mathcal{F} = -1.3 \cdot 10^{-3}\,\mathrm{Nm}^{-2}$.

The Casimir effect was first predicted in 1948 [H.G.B. Casimir, Proc. K. Ned. Akad. Wet. **51**, 793 (1948)], but successfully confirmed only as late as 1958 [M.J. Spaarnay, Physica **24**, 751 (1958)]. In this experiment, two parallel plates of lateral size $L = 5$ cm were held $1\,\mu$m apart which, by Eq. (3.13), results in an attractive force of $F = -3.3\,\mu$N which is tiny, indeed. Current high-precision experiments are performed with experimental set-ups such as that depicted in Fig. 3.

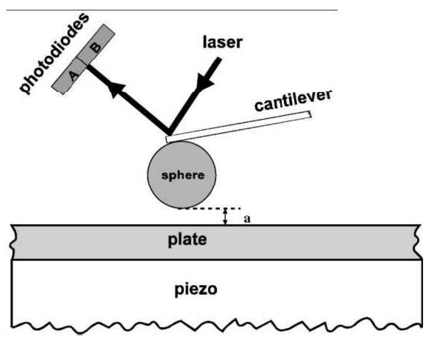

FIG. 3: Experimental set-up for measuring Casimir forces between a sphere and a plate. Applying a voltage to the piezo changes the distance a between the two objects.

Further reading on the Casimir and related effects:

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