## Lecture 20: Cavity QED — the Jaynes–Cummings model

- Jaynes–Cummings Hamiltonian, dressed states
- Collapse and revival of atomic populations

Jaynes–Cummings Hamiltonian, dressed states: We now turn away from the atom in free space and consider an atom in a high-quality resonator. We assume that the quality of the resonator is high enough so that the atom effectively only interacts with a single mode of the cavity field. The Hamiltonian (15.6) can then be written as

$$\hat{H} = \hbar \omega \hat{a}^{\dagger} \hat{a} + \frac{1}{2} \hbar \omega_A \hat{\sigma}_z - \hbar g (\hat{a} \hat{\sigma}^{\dagger} + \hat{a}^{\dagger} \hat{\sigma}) , \qquad (20.1)$$

where the coupling constant g is effectively the product of the dipole moment and the mode function of the cavity mode at the position of the atom.

The Hamiltonian (20.1) describes one of the few exactly solvable models in quantum optics. Since the interaction Hamiltonian only couples pairs of atom-field states  $\{|n+1,g\rangle, |n,e\rangle\}$ , the Jaynes–Cummings Hamiltonian (20.1) decouples into an infinite direct product of 2 × 2-matrix Hamiltonians

$$\hat{H}_n \begin{pmatrix} |n+1,g\rangle \\ |n,e\rangle \end{pmatrix} = \hbar \begin{pmatrix} (n+1)\omega - \frac{1}{2}\omega_A & -g\sqrt{n+1} \\ -g\sqrt{n+1} & n\omega + \frac{1}{2}\omega_A \end{pmatrix} \begin{pmatrix} |n+1,g\rangle \\ |n,e\rangle \end{pmatrix}.$$
 (20.2)

The eigenvalue problem for this Hamiltonian yields the eigenfrequencies

$$\omega_{n,\pm} = \left(n + \frac{1}{2}\right)\omega \pm \frac{1}{2}\sqrt{\delta^2 + \Omega_n^2} \tag{20.3}$$

where, as previously,  $\delta = \omega - \omega_A$  is the detuning of the radiation frequency from the atomic resonance. The parameter  $\Omega_n = 2g\sqrt{n+1}$  is called the *n*-photon Rabi frequency. The corresponding eigenstates are

$$|n, +\rangle = \cos \Theta_n |n+1, g\rangle - \sin \Theta_n |n, e\rangle,$$
  
$$|n, -\rangle = \sin \Theta_n |n+1, g\rangle + \cos \Theta_n |n, e\rangle,$$
 (20.4)

where  $[\Delta_n = \sqrt{\delta^2 + \Omega_n^2}]$ 

$$\cos \Theta_n = \frac{\Delta_n - \delta}{\sqrt{(\Delta_n - \delta)^2 + \Omega_n^2}}, \quad \sin \Theta_n = \frac{\Omega_n}{\sqrt{(\Delta_n - \delta)^2 + \Omega_n^2}}.$$
 (20.5)

The eigenstates  $|n, \pm\rangle$  are called *dressed-atom* states. The unperturbed atomic eigenstates  $|g\rangle$  and  $|e\rangle$  are modified (dressed) by the interaction with the cavity field, and their eigenfrequencies are shifted by an amount determined by the coupling strength. This is the dynamical Stark effect. For zero detuning,  $\delta = 0$ , the unperturbed (degenerate) eigenfrequencies of the states  $|n+1,g\rangle$  and  $|n,e\rangle$  are  $(n+1/2)\omega$  which are split due to the interaction by an amount  $\Omega_n = 2g\sqrt{n+1}$  (see Fig. 25). [Note that a similar level shift, the Lamb shift, appears for an atom in the weak-coupling regime such as in free space.] Even if no photon is present in the cavity, there will be a level splitting, the *vacuum Rabi splitting*  $\Omega_0 = 2g$ , between the exact eigenstates of the combined atom-cavity system.

Since we know the eigenvalues and eigenstates of the Jaynes–Cummings Hamiltonian (20.1), we know how to compute the unitary time evolution operator  $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$ . The dressed states (20.4), together with the remaining uncoupled state  $|0,g\rangle$ , form a complete set of eigenstates of the Hamiltonian. Hence, the unitary operator  $\hat{U}$  can be written as

$$\hat{U}(t) = e^{-i\frac{\hat{H}}{\hbar}t} = e^{i\frac{\omega_{A}}{2}t}|0,g\rangle\langle0,g| + \sum_{\sigma=\pm}^{\infty}\sum_{n=0}^{\infty}e^{-i\omega_{n,\sigma}t}|n,\sigma\rangle\langle n,\sigma|$$

$$= e^{i\frac{\omega_{A}}{2}t}|0,g\rangle\langle0,g| + \sum_{n=0}^{\infty}e^{-i(n+\frac{1}{2})\omega t}\left[e^{-i\frac{\Delta_{n}}{2}t}|n,+\rangle\langle n,+|+e^{i\frac{\Delta_{n}}{2}t}|n,-\rangle\langle n,-|\right]$$

$$= e^{i\frac{\omega_{A}}{2}t}|0,g\rangle\langle0,g| + \sum_{n=0}^{\infty}e^{-i(n+\frac{1}{2})\omega t}$$

$$\times \left\{\left[\cos\frac{\Delta_{n}t}{2} + i\frac{\delta}{\Delta_{n}}\sin\frac{\Delta_{n}t}{2}\right]|n+1,g\rangle\langle n+1,g| + \left[\cos\frac{\Delta_{n}t}{2} - i\frac{\delta}{\Delta_{n}}\sin\frac{\Delta_{n}t}{2}\right]|n,e\rangle\langle n,e| + i\frac{\Omega_{n}}{\Delta_{n}}\sin\frac{\Delta_{n}t}{2}\left(|n+1,g\rangle\langle n,e|+|n,e\rangle\langle n+1,g|\right)\right\}$$
(20.6)

where we used the expressions (20.4) to write the dressed states in terms of the unperturbed eigenstates.

The unitary operator (20.6) describes the full dynamics of the Jaynes–Cummings model. In particular, we can compute the time evolution of the density operator

$$\hat{\varrho}(t) = \hat{U}(t)\hat{\varrho}(0)\hat{U}^{\dagger}(t)$$
. (20.7)

Let the density operator at the initial time be in a product state,  $\hat{\varrho}(0) = \hat{\varrho}_F(0) \otimes \hat{\sigma}(0)$ . The quantum state of the atom alone is then obtained by taking the trace over the photonic

degrees of freedom,

$$\hat{\sigma}(t) = \operatorname{Tr}_{F}\hat{\varrho}(t) = \operatorname{Tr}_{F}\left[\hat{U}(t)\left(\hat{\varrho}_{F}(0)\otimes\hat{\sigma}(0)\right)\hat{U}^{\dagger}(t)\right] = \sum_{n=0}^{\infty} \langle n|\hat{U}(t)\left(\hat{\varrho}_{F}(0)\otimes\hat{\sigma}(0)\right)\hat{U}^{\dagger}(t)|n\rangle.$$
(20.8)

Let us concentrate on the atomic excited-state population  $\sigma_{ee}(t)$ . Note that this an experimentally accessible quantity which can be measured by ionizing the atom stateselectively. If a sample of equally prepared atoms is sent through the cavity and, after having left the cavity, is irradiated with a laser of a frequency that just exceeds the ionization energy of the excited state, the recorded fraction of ions out of the total atom number is just  $\sigma_{ee}(t)$ .

The excited-state population can be computed from Eq. (20.8) as

$$\sigma_{ee}(t) = \sum_{n=0}^{\infty} \langle n, e | \hat{U}(t) \left( \hat{\varrho}_F(0) \otimes \hat{\sigma}(0) \right) \hat{U}^{\dagger}(t) | n, e \rangle .$$
(20.9)

The general expression (20.9) with  $\hat{U}(t)$  given by Eq. (20.6) is of rather complicated form which simplifies considerably in the case of zero detuning,  $\delta = 0$ ,

$$\sigma_{ee}(t) = \sum_{n=0}^{\infty} \left\{ \cos^2 \frac{\Omega_n t}{2} \varrho_{n,n} \sigma_{ee} + \sin^2 \frac{\Omega_n t}{2} \varrho_{n+1,n+1} \sigma_{gg} + \operatorname{Im} \left[ \frac{1}{2} \sin \Omega_n t \, \varrho_{n+1,n} \sigma_{ge} \right] \right\} \,.$$
(20.10)

If the atom has been initially prepared in its excited state,  $\sigma_{ee}(0) = 1$ , then the time evolution simplifies to

$$\sigma_{ee}(t) = \frac{1}{2} \left[ 1 + \sum_{n=0}^{\infty} \cos \Omega_n t \, \varrho_{n,n}(0) \right] \,. \tag{20.11}$$

**Collapse and revival of atomic populations**: The atomic excited-state population, Eq. (20.11), is an incoherent sum of oscillations with the *n*-photon Rabi frequencies  $\Omega_n = 2g\sqrt{n+1}$ . Only if the cavity field had initially been prepared in a number state  $|k\rangle$ , the atomic population oscillates as a single sinusoidal function,  $\sigma_{ee}(t) = (1 + \cos \Omega_k t)/2$ . This behaviour should be compared with the time evolution of a resonantly driven twolevel atom in free space, Eq. (19.7), which in the absence of spontaneous decay, reduces to  $\sigma_{ee}(t) = (1 + \cos \Omega t)/2$ . But there the radiation field was assumed to be prepared in a coherent state.

The incoherent summation of oscillatory terms with incommensurable frequencies in Eq. (20.11) seems to wash out all the oscillations if the cavity field contains more than one number state contribution. A coherent state with amplitude  $\alpha$  shows a Poissonian



FIG. 25: Resonant atom-light interaction ( $\delta = 0$ ) lifts the degeneracy of the unperturbed states  $\{|n, e\rangle, |n + 1, g\rangle\}$ . The level splitting  $\Omega_n = 2g\sqrt{n+1}$  depends on the number of photons.



FIG. 26: Time evolution of the excited-state population  $\sigma_{ee}(t)$  with the cavity field in a coherent state with mean photon number  $|\alpha|^2 = 10$ .

distribution of photon numbers, Eq. (5.10),  $\rho_{n,n}(0) = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}$ , in which case

$$\sigma_{ee}(t) = \frac{1}{2} \left[ 1 + \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} \cos(2g\sqrt{n+1}t) \right] \,. \tag{20.12}$$

Figure 26 shows the time evolution of the excited-state population (20.12) for  $|\alpha|^2 = 10$ . The oscillations collapse after a certain time, but re-appear periodically. This *collapse and revival* is a quantum interference effect and has nothing to do with dissipation as in resonance fluorescence.

For large mean photon number,  $|\alpha|^2 \gg 1$ , the Rabi frequencies can be approximated by

$$\Omega_n \approx 2g\sqrt{|\alpha|^2 + 1} \left( 1 + \frac{n - |\alpha|^2}{2(|\alpha|^2 + 1)} + \cdots \right) \approx 2g|\alpha| \left( 1 + \frac{n - |\alpha|^2}{2|\alpha|^2} \right).$$
(20.13)

With that, we can perform the summation analytically to obtain

$$\sigma_{ee}(t) \simeq \frac{1}{2} \left[ 1 + e^{-|\alpha|^2} \operatorname{Re} e^{ig|\alpha|t} e^{|\alpha|^2 \exp(igt/|\alpha|)} \right].$$
(20.14)

For sufficiently short times such that  $gt \ll |\alpha|$ , we can expand the double exponential to second order and are left with

$$\sigma_{ee}(t) \simeq \frac{1}{2} \left[ 1 + e^{-g^2 t^2/2} \cos 2g |\alpha| t \right]$$
(20.15)

which shows that the oscillations with an effective Rabi frequency  $\Omega_{\text{eff}} = 2g|\alpha|$  collapse after a characteristic time  $t_{\text{collapse}} = \sqrt{2}/g$ .

## Lecture 21: Application of cavity QED: Entangled-state generation

- Generation of atomic Bell states
- Dispersive limit of the Jaynes–Cummings model
- Generation of entangled states between atoms and the cavity field

<u>Generation of atomic Bell states</u>: Cavity QED is an important tool to generate and understand nonclassical states of light and atoms, in particular entangled states between either an atom and the cavity field or between different atoms. Let us start with an example where two atoms can be entangled to form a maximally entangled state  $|\Psi^-\rangle$ , a so-called Bell state. Let the initial state of the two atoms and the cavity field be

$$|\Psi(0)\rangle = |e\rangle_1 |g\rangle_2 |0\rangle, \qquad (21.1)$$

that is, the first atom to enter the initially empty cavity is prepared in its excited state  $|e\rangle_2$ , and the second atom, which is assumed to enter the cavity only after the first atom has left, in prepared in its ground state  $|g\rangle$ .

If the first atom interacts with the cavity on resonance, we find by using Eq. (20.6) that

$$\hat{U}(t)|\Psi(0)\rangle = e^{-\frac{i}{2}\omega t} \left[\cos\frac{\Omega_0 t}{2}|0\rangle|e\rangle_1 + i\sin\frac{\Omega_0 t}{2}|1\rangle|g\rangle_1\right]|g\rangle_2.$$
(21.2)

After a time  $t_1 = \pi/(2\Omega_0)$ , the state of the atom-cavity system will be

$$|\Psi(t_1)\rangle = e^{-i\omega/(4\Omega_0)} \frac{1}{\sqrt{2}} \Big[ |0\rangle|e\rangle_1 + i|1\rangle|g\rangle_1 \Big] |g\rangle_2 \,. \tag{21.3}$$

That is, the first atom and the cavity are in an equal superposition of the initial state and the atom having deposited a photon in the cavity.

In the second step, the second atom interacts with the cavity field that contains either zero or one photon. Hence, the atom can either pick the photon up or leave the cavity in its ground state. The interaction

$$\hat{U}(t)|\Psi(t_1)\rangle = \frac{e^{-i\pi\omega/(4\Omega_0)}}{\sqrt{2}} \left[ e^{\frac{i}{2}\omega t}|0\rangle|e\rangle_1|g\rangle_2 + e^{-\frac{i}{2}\omega t} \left(\cos\frac{\Omega_0 t}{2}|1\rangle|g\rangle_1|g\rangle_2 + i\sin\frac{\Omega_0 t}{2}|0\rangle|g\rangle_1|e\rangle_2 \right) \right]$$

turns, after an interaction time  $t_2 = 2t_1 = \pi/\Omega_0$ , the state  $|\Psi(t_1)\rangle$  into

$$|\Psi(t_1+t_2)\rangle = \hat{U}(t_2)|\Psi(t_1)\rangle = e^{-i\pi\omega/(4\Omega_0)} \frac{1}{\sqrt{2}} \Big[ e^{i\pi\omega/(2\Omega_0)} |e\rangle_1 |g\rangle_2 - e^{-i\pi\omega/(2\Omega_0)} |g\rangle_1 |e\rangle_2 \Big] |0\rangle.$$

Taking out the free evolution, i.e. in the interaction picture, the state is simplified to

$$|\Psi(t_1+t_2)\rangle = \frac{1}{\sqrt{2}} \Big[ |e\rangle_1 |g\rangle_2 - |g\rangle_1 |e\rangle_2 \Big] |0\rangle.$$
(21.4)

The state (21.4) is now indeed an atomic Bell state  $|\Psi^{-}\rangle$ , i.e. a maximally entangled state. Note that the state of the two atoms is decoupled from the quantum state of the cavity. The cavity plays merely the role of an agent or catalyst to achieve the effective interaction between the two atoms (Fig. 27).



FIG. 27: Schematic experimental apparatus used to create atomic Bell states [after E. Hagley *et al.*, Phys. Rev. Lett. **79**, 1 (1997).]

Dispersive limit of the Jaynes–Cummings model: Instead of entangling two atoms, one can use cavity QED to entangle one atom with the cavity field in a coherent state. For that, let us look at the dispersive limit of the Jaynes–Cummings model in which the cavity resonance is far detuned from the atomic resonance,  $\delta \gg \Omega_n$ . In this limit,  $\Delta_n \simeq \delta + \Omega_n^2/(2\delta) = \delta + 2g^2(n+1)/\delta$ , and the dressed states (20.4) are approximately the unperturbed eigenstates,  $|n, +\rangle \simeq |n, e\rangle$  and  $|n, -\rangle \simeq |n+1, g\rangle$ .

With these approximations, the unitary evolution operator  $\hat{U}(t)$  [Eq. (20.6)] of the Jaynes–Cummings model can be written as

$$\hat{U}(t) \simeq e^{\frac{i}{2}\omega_{A}t}|0,g\rangle\langle0,g| + \sum_{n=0}^{\infty} e^{-i(n+\frac{1}{2})\omega t} \\ \times \left[e^{-i(\frac{\delta}{2}+\frac{g^{2}}{\delta}(n+1))t}|n,e\rangle\langle n,e| + e^{i(\frac{\delta}{2}+\frac{g^{2}}{\delta}(n+1))t}|n+1,g\rangle\langle n+1,g|\right].$$
(21.5)

This can be equivalently rewritten in operator form using the free Hamiltonian  $\hat{H}_0 = \hbar \omega \hat{n} + \frac{1}{2}\hbar \omega_A \hat{\sigma}_z$  as

$$\hat{U}(t) = \exp\left(-\frac{i}{\hbar}\hat{H}_0t\right) \left[\exp\left(-i\frac{g^2}{\delta}(\hat{n}+1)t\right)|e\rangle\langle e| + \exp\left(i\frac{g^2}{\delta}\hat{n}t\right)|g\rangle\langle g|\right].$$
(21.6)

The effective unitary operator (21.6) is diagonal in the basis of the unperturbed eigenstates and describes conditional phase shifts depending on the atomic state.

Generation of entangled states between atoms and the cavity field: The statedependence of the phase shift can be advantageously used to generate nonclassical states. In particular, entangled states of atoms and light are easily produced. Let us consider first the situation in which a single atom has been prepared in a superposition of its ground and excited states. This preparation can be achieved by applying a  $\pi/2$ -pulse in either free space or a low-quality cavity (see discussion on Rabi flopping in a previous lecture). Assuming that the cavity field is prepared in a coherent state with complex amplitude  $\alpha$ , the initial state of the atom-cavity system is

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} \left(|g\rangle + |e\rangle\right) |\alpha\rangle.$$
(21.7)



FIG. 28: Schematic experimental apparatus used to create Schrödinger cat states between a Rydberg atom and the cavity field [after M. Brune *et al.*, Phys. Rev. Lett. **77**, 4887 (1996).]

Next, let us examine the effect of an operator of the type  $e^{i\Phi\hat{n}}$  on coherent states. Recall that the coherent states can be expanded in terms of number states as

$$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-\frac{|\alpha|^2}{2}} |n\rangle \,.$$

Therefore,

$$e^{i\Phi\hat{n}}|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n e^{i\Phi n}}{\sqrt{n!}} e^{-\frac{|\alpha|^2}{2}} |n\rangle = |\alpha e^{i\Phi}\rangle.$$
(21.8)

If we let  $\Phi = g^2 t / \delta$ , the action of the dispersive interaction (21.6) on the initial state (21.7) yields

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle = \exp\left(-\frac{i}{\hbar}\hat{H}_0t\right)\frac{1}{\sqrt{2}}\Big[|g\rangle|\alpha e^{i\Phi}\rangle + e^{-i\Phi}|e\rangle|\alpha e^{-i\Phi}\rangle\Big].$$
(21.9)

The state (21.9) is an entangled state between the atom being in its ground state and the coherent state being rotated by an angle  $+\Phi$ , and the atom in its excited state and the coherent state rotated by the opposite angle  $-\Phi$  (see Fig. 29). It is an example of a type of quantum state infamously dubbed '*Schrödinger's cat*' which is a quantum superposition of a microscopic (atomic) system and a macroscopic or mesoscopic (photonic) system (the 'cat') and first served as the prime example of an entangled state. The type of quantum states



FIG. 29: Photonic part of the Wigner function. The initial state is depicted as a full circle, the two parts forming the quantum superposition in open circles.

have been realized experimentally by using Rydberg atoms (highly excited atoms with very high radial quantum numbers  $n \gtrsim 60$ ). Figure 28 shows the schematic experimental set-up that was used to create Schrödinger cat states between a Rydberg atom and the cavity field. The main cavity is denoted by C.  $R_1$  and  $R_2$  are two Ramsey zones that are used to apply  $\pi/2$ -pulses to the atom.