## Lecture 1: Classical electrodynamics - Maxwell's equations

- Maxwell's equations in vacuum, potentials, Coulomb gauge
- Mode decomposition

Maxwell's equations in vacuum: Quantum optics is, as its name suggests, the quantum theory of light. In order to describe the wave properties of light, we first need to look at Maxwell's equations. The classical electromagnetic field in free space is described by the following set of equations:

$$
\begin{array}{rlr}
\boldsymbol{\nabla} \cdot \mathbf{B}(\mathbf{r}, t)=0 & (\text { Gauß' law) }, \\
\boldsymbol{\nabla} \cdot \mathbf{D}(\mathbf{r}, t)=0 & \text { (Coulomb's law), } \\
\boldsymbol{\nabla} \times \mathbf{E}(\mathbf{r}, t)=-\dot{\mathbf{B}}(\mathbf{r}, t) & (\text { Faraday's law) }, \\
\boldsymbol{\nabla} \times \mathbf{H}(\mathbf{r}, t) & =\dot{\mathbf{D}}(\mathbf{r}, t) & \text { (Amperé's law) } . \tag{1.4}
\end{array}
$$

In order to fix terminology, we will call the fields appearing in Eqs. (1.1)-(1.4) the electric field $[\mathbf{E}(\mathbf{r}, t)]$, the displacement field $[\mathbf{D}(\mathbf{r}, t)]$, the induction field $[\mathbf{B}(\mathbf{r}, t)]$, and the magnetic field $[\mathbf{H}(\mathbf{r}, t)]$. These fields are not independent of each other. They are related by some constitutive relations, which for the electromagnetic field in free space read

$$
\begin{align*}
& \mathbf{D}(\mathbf{r}, t)=\varepsilon_{0} \mathbf{E}(\mathbf{r}, t),  \tag{1.5}\\
& \mathbf{H}(\mathbf{r}, t)=\frac{1}{\mu_{0}} \mathbf{B}(\mathbf{r}, t) . \tag{1.6}
\end{align*}
$$

Here, $\varepsilon_{0}$ and $\mu_{0}$ are the vacuum permittivity and permeability, respectively. The are related to the speed of light $c$ by $\varepsilon_{0} \mu_{0}=1 / c^{2}$.

Vector and scalar potentials: Our first task will be to solve Maxwell's equations (1.1) (1.4). This is generally done by introducing a vector potential $\mathbf{A}(\mathbf{r}, t)$ and a scalar potential $\phi(\mathbf{r}, t)$. Recall that there are two important vector identities:

$$
\begin{align*}
\boldsymbol{\nabla} \cdot(\boldsymbol{\nabla} \times \mathbf{V}) & \equiv 0 \forall \text { vector functions } \mathbf{V}(\text { Gauß' theorem }),  \tag{1.7}\\
\boldsymbol{\nabla} \times(\boldsymbol{\nabla} S) & \equiv 0 \forall \text { scalar functions } S \text { (Stokes' theorem) } . \tag{1.8}
\end{align*}
$$

Gauß' theorem, Eq. (1.7), can be used to solve Eq. (1.1) by introducing the vector potential as

$$
\begin{equation*}
\mathbf{B}(\mathbf{r}, t)=\boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r}, t) . \tag{1.9}
\end{equation*}
$$

Remember that this is just an identity. Because of Gauß' theorem, the induction field can always be written as the curl of some other function. If we now insert the definition (1.9) into Faraday's law (1.3) and write it as $\boldsymbol{\nabla} \times[\mathbf{E}(\mathbf{r}, t)+\dot{\mathbf{A}}(\mathbf{r}, t)]=0$, we can use Stokes' theorem (1.8) and define the scalar potential as

$$
\begin{equation*}
\mathbf{E}(\mathbf{r}, t)=-\dot{\mathbf{A}}(\mathbf{r}, t)-\nabla \phi(\mathbf{r}, t) . \tag{1.10}
\end{equation*}
$$

Because the introduction of the potentials is just a mathematical trick to simplify Maxwell's equations, the potentials have no physical meaning whatsoever. The physical quantities are always the electromagnetic fields.

There are two more of Maxwell's equations to satisfy, Coulomb's law (1.2) and Amperé's law (1.4). Using the constitutive relations (1.5) and (1.6), we obtain the following wave equations for the potentials:

$$
\begin{align*}
\boldsymbol{\nabla} \cdot \mathbf{D}(\mathbf{r}, t)=0 & \Rightarrow-\varepsilon_{0}[\Delta \phi(\mathbf{r}, t)+\boldsymbol{\nabla} \cdot \dot{\mathbf{A}}(\mathbf{r}, t)]=0,  \tag{1.11}\\
\nabla \times \mathbf{H}(\mathbf{r}, t)=\dot{\mathbf{D}}(\mathbf{r}, t) & \Rightarrow \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r}, t)+\frac{1}{c^{2}} \ddot{\mathbf{A}}(\mathbf{r}, t)=-\frac{1}{c^{2}} \boldsymbol{\nabla} \dot{\phi}(\mathbf{r}, t) . \tag{1.12}
\end{align*}
$$

The name wave equation will become clear shortly.
Gauge freedom: The two wave equations are still very complicated, because they form a coupled set of partial differential equations that is very hard to solve. But they can be substantially simplified by noting that the potentials are actually not uniquely defined. The definition of the vector potential (1.9) and Stokes' theorem (1.8) imply that the transformed vector potential $\mathbf{A}^{\prime}(\mathbf{r}, t)=\mathbf{A}(\mathbf{r}, t)+\boldsymbol{\nabla} g(\mathbf{r}, t)$, where $g(\mathbf{r}, t)$ is an arbitrary scalar function, does not change the induction field $\mathbf{B}(\mathbf{r}, t)$. If we insert the transformed vector potential into Eq. (1.10), we see that we have to transform the scalar potential simultaneously as $\phi^{\prime}(\mathbf{r}, t)=\phi(\mathbf{r}, t)-\dot{g}(\mathbf{r}, t)$ in order not to change the electric field $\mathbf{E}(\mathbf{r}, t)$. This freedom in choosing a function $g(\mathbf{r}, t)$ is called gauge freedom.

Coulomb gauge: In quantum optics, one frequently uses the Coulomb gauge where one chooses a gauge function $g(\mathbf{r}, t)$ such that

$$
\begin{equation*}
\boldsymbol{\nabla} \cdot \mathbf{A}(\mathbf{r}, t)=0 . \tag{1.13}
\end{equation*}
$$

This condition states that the vector potential is a transverse vector function. One can see that easily by Fourier transforming Eq. (1.13) to

$$
\begin{equation*}
i \mathbf{k} \cdot \tilde{\mathbf{A}}(\mathbf{k}, t)=0 \tag{1.14}
\end{equation*}
$$

which is the familiar transversality condition for a wave with wavevector $\mathbf{k}$ (see Fig. 1). Moreover, because of Eq. (1.11), the scalar potential vanishes in the Coulomb gauge. The


FIG. 1: In the Coulomb gauge, $\mathbf{E}$ and $\mathbf{B}$ are both perpendicular to the propagation direction.
second wave equation, Eq. (1.12), then leads to the wave equation for the vector potential

$$
\begin{equation*}
\Delta \mathbf{A}(\mathbf{r}, t)-\frac{1}{c^{2}} \ddot{\mathbf{A}}(\mathbf{r}, t)=\mathbf{0} \text {. } \tag{1.15}
\end{equation*}
$$

Here we have used the vector identity $\boldsymbol{\nabla} \times(\boldsymbol{\nabla} \times \mathbf{V})=\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{V})-\Delta \mathbf{V}$.

Add-on: Another frequently used gauge is the Lorenz gauge where a gauge function $g(\mathbf{r}, t)$ is chosen such that

$$
\frac{1}{c^{2}} \dot{\phi}(\mathbf{r}, t)+\boldsymbol{\nabla} \cdot \mathbf{A}(\mathbf{r}, t)=0
$$

This gauge has the advantage that is preserves the relativistic invariance of Maxwell's equations. If we define the contravariant four-potential as $A^{\mu}=(\phi / c, \mathbf{A})$, this gauge condition can be written as $\partial_{\mu} A^{\mu}=0\left[\partial_{\mu}=\right.$ $\left.\left(\partial_{t} / c, \boldsymbol{\nabla}\right)\right]$. The wave equations (1.11) and (1.12) then decouple as

$$
\Delta \phi(\mathbf{r}, t)-\frac{1}{c^{2}} \ddot{\phi}(\mathbf{r}, t)=0, \quad \Delta \mathbf{A}(\mathbf{r}, t)-\frac{1}{c^{2}} \ddot{\mathbf{A}}(\mathbf{r}, t)=\mathbf{0}
$$

which can be written in four-vector notation as$A^{\mu}=0$ $\square$ $\left.=\partial_{\nu} \partial^{\nu}\right]$.

Because quantum optics is a non-relativistic theory, we do not need relativistic invariance.

Hence, from now on we will use the Coulomb gauge (1.13) in which the vector potential is transverse and satisfies the wave equation (1.15).

Mode decomposition: The partial differential equation (1.15) obeyed by the vector potential is solved by separation of variables, or mode decomposition, in which we seek to separate the spatial dependence of the vector potential from its time dependence. In order to do so, we rewrite Eq. (1.15) with a separation constant $\omega_{\lambda}^{2} / c^{2}$ as

$$
\begin{equation*}
\left[\left(\Delta+\frac{\omega_{\lambda}^{2}}{c^{2}}\right)-\left(\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}}+\frac{\omega_{\lambda}^{2}}{c^{2}}\right)\right] \mathbf{A}(\mathbf{r}, t)=\mathbf{0} \tag{1.16}
\end{equation*}
$$

and make a separation ansatz for the vector potential as

$$
\begin{equation*}
\mathbf{A}(\mathbf{r}, t)=\sum_{\lambda} \mathbf{A}_{\lambda}(\mathbf{r}) u_{\lambda}(t) . \tag{1.17}
\end{equation*}
$$

Mode functions: The mode functions $\mathbf{A}_{\lambda}(\mathbf{r})$ are solutions to the scalar Helmholtz equation

$$
\begin{equation*}
\left(\Delta+\frac{\omega_{\lambda}^{2}}{c^{2}}\right) \mathbf{A}_{\lambda}(\mathbf{r})=\mathbf{0} . \tag{1.18}
\end{equation*}
$$

If we make the identification $k^{2}=\omega_{\lambda}^{2} / c^{2}$, we see that the possible solutions to the Helmholtz equation in cartesian co-ordinates are plane waves $e^{i \mathbf{k} \cdot \mathbf{r}}$. For every wave vector $\mathbf{k}$, there are two possible orthogonal polarizations with unit vectors $\mathbf{e}_{\sigma}$ such that $\mathbf{e}_{\sigma} \cdot \mathbf{k}=0$. Hence, the sum over $\lambda$ has in fact the following meaning: $\sum_{\lambda} \equiv \sum_{\sigma=1}^{2} \int d^{3} k$. Similarly, in other coordinate systems, we obtain solutions in terms of (cylindrical or spherical) Bessel functions. The possible solutions to the scalar Helmholtz equation and the meaning of the mode sum are collected in the following table:

$$
\begin{gathered}
(x, y, z): \lambda \equiv(\sigma, \mathbf{k}), \quad \mathbf{A}_{\lambda}(\mathbf{r}) \equiv \mathbf{e}_{\sigma} e^{i \mathbf{k} \cdot \mathbf{r}}, \quad \sum_{\lambda} \equiv \sum_{\sigma=1}^{2} \int d^{3} k \\
(\rho, \phi, z): \lambda \equiv\left(\sigma, n, k_{\rho}, k_{z}\right), \quad \mathbf{A}_{\lambda}(\mathbf{r}) \equiv \mathbf{e}_{\sigma} J_{n}\left(k_{\rho} \rho\right) e^{i n \phi+i k_{z} z}, \sum_{\lambda} \equiv \sum_{\sigma=1}^{2} \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} d k_{\rho} \int_{-\infty}^{\infty} d k_{z} \\
(\rho, \theta, \phi): \lambda \equiv\left(\sigma, k_{\rho}, l, m\right), \quad \mathbf{A}_{\lambda}(\mathbf{r}) \equiv \mathbf{e}_{\sigma} j_{l}\left(k_{\rho} \rho\right) Y_{l m}(\theta, \phi), \sum_{\lambda} \equiv \sum_{\sigma=1}^{2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_{0}^{\infty} d k_{\rho}
\end{gathered}
$$

The temporal part of the Helmholtz equation reduces to the equation

$$
\begin{equation*}
\ddot{u}_{\lambda}(t)+\omega_{\lambda}^{2} u_{\lambda}(t)=0, \tag{1.19}
\end{equation*}
$$

which is the differential equation for a harmonic oscillator with frequency $\omega_{\lambda}$, with the solutions $u_{\lambda}(t)=e^{ \pm i \omega_{\lambda} t} u_{\lambda}(0)$. This finally explains why the partial differential equation satisfied by the vector potential is called a wave equation because its solutions are either plane waves or Bessel waves.

## Lecture 2: Quantization of the free electromagnetic field

- classical Hamiltonian, harmonic oscillators
- photon creation and annihilation operators

Classical Hamiltonian: We will now use the mode decomposition of the classical vector potential to find out its consequences for canonical quantization. For this purpose, we will focus on the mode decomposition in cartesian coordinates,

$$
\begin{equation*}
\mathbf{A}(\mathbf{r}, t)=\sum_{\sigma} \int \frac{d^{3} k}{(2 \pi)^{3 / 2}} \mathbf{e}_{\sigma} e^{i \mathbf{k} \cdot \mathbf{r}} u_{\mathbf{k} \sigma}(t) . \tag{2.1}
\end{equation*}
$$

In order to ensure that the vector potential is real, we add its complex conjugate with the condition $u_{\mathbf{k} \sigma}(t)=u_{-\mathbf{k} \sigma}^{*}(t)$, and obtain

$$
\begin{equation*}
\mathbf{A}(\mathbf{r}, t)=\sum_{\sigma} \int \frac{d^{3} k}{(2 \pi)^{3 / 2}} \mathbf{e}_{\sigma}\left[u_{\mathbf{k} \sigma} e^{i(\mathbf{k} \cdot \mathbf{r}-\omega t)}+u_{\mathbf{k} \sigma}^{*} e^{-i(\mathbf{k} \cdot \mathbf{r}-\omega t)}\right] \tag{2.2}
\end{equation*}
$$

From classical electromagnetism we know that the energy stored in the electromagnetic field, i.e. the classical Hamiltonian function, is

$$
\begin{equation*}
H=\frac{1}{2} \int d^{3} r\left[\varepsilon_{0} \mathbf{E}^{2}(\mathbf{r}, t)+\frac{1}{\mu_{0}} \mathbf{B}^{2}(\mathbf{r}, t)\right], \tag{2.3}
\end{equation*}
$$

where we have to express the electric field and the magnetic induction in terms of the vector potential. Inserting (2.2) into the Hamiltonian (2.3), we find that

$$
\begin{align*}
H & =-\frac{1}{2} \sum_{\sigma, \sigma^{\prime}} \iint \frac{d^{3} r d^{3} k d^{3} k^{\prime}}{(2 \pi)^{3}}\left\{\left[\varepsilon_{0}\left(\mathbf{e}_{\sigma} \cdot \mathbf{e}_{\sigma^{\prime}}\right) \omega \omega^{\prime}+\frac{1}{\mu_{0}}\left(\mathbf{k} \times \mathbf{e}_{\sigma}\right) \cdot\left(\mathbf{k}^{\prime} \times \mathbf{e}_{\sigma^{\prime}}\right)\right]\right. \\
& \left.\times\left[u_{\mathbf{k} \sigma} e^{i(\mathbf{k} \cdot \mathbf{r}-\omega t)}-u_{\mathbf{k} \sigma}^{*} e^{-i(\mathbf{k} \cdot \mathbf{r}-\omega t)}\right]\left[u_{\mathbf{k}^{\prime} \sigma^{\prime}} e^{i\left(\mathbf{k}^{\prime} \cdot \mathbf{r}-\omega^{\prime} t\right)}-u_{\mathbf{k}^{\prime} \sigma^{\prime}}^{*} e^{-i\left(\mathbf{k}^{\prime} \cdot \mathbf{r}-\omega^{\prime} t\right)}\right]\right\} . \tag{2.4}
\end{align*}
$$

Using the orthogonality of the polarization vectors $\mathbf{e}_{\sigma} \cdot \mathbf{e}_{\sigma^{\prime}}=\delta_{\sigma, \sigma^{\prime}}$ and the relation $\left(\mathbf{k} \times \mathbf{e}_{\sigma}\right)$. $\left(\mathbf{k} \times \mathbf{e}_{\sigma^{\prime}}\right)=k^{2}\left(\mathbf{e}_{\sigma} \cdot \mathbf{e}_{\sigma^{\prime}}\right)$, and integrating over $\mathbf{r}$ and $\mathbf{k}^{\prime}$ yields

$$
\begin{equation*}
H=2 \varepsilon_{0} \sum_{\sigma} \int d^{3} k \omega^{2}\left|u_{\mathbf{k} \sigma}\right|^{2} . \tag{2.5}
\end{equation*}
$$

Because the expansion coefficients $u_{\mathbf{k} \sigma}$ are complex functions, we can split them into their respective real and imaginary parts,

$$
\left.\begin{array}{l}
u_{\mathbf{k} \sigma}=\frac{1}{2 \sqrt{\varepsilon_{0}}}\left[q_{\mathbf{k} \sigma}+i \frac{p_{\mathbf{k} \sigma}}{\omega}\right],  \tag{2.6}\\
u_{\mathbf{k} \sigma}^{*}=\frac{1}{2 \sqrt{\varepsilon_{0}}}\left[q_{\mathbf{k} \sigma}-i \frac{p_{\mathbf{k} \sigma}}{\omega}\right],
\end{array}\right\} \begin{aligned}
& q_{\mathbf{k} \sigma}=\sqrt{\varepsilon_{0}}\left(u_{\mathbf{k} \sigma}+u_{\mathbf{k} \sigma}^{*}\right), \\
& p_{\mathbf{k} \sigma}=-i \omega \sqrt{\varepsilon_{0}}\left(u_{\mathbf{k} \sigma}-u_{\mathbf{k} \sigma}^{*}\right) .
\end{aligned}
$$

With these definitions, we find that the classical Hamiltonian can be rewritten as

$$
\begin{equation*}
H=\frac{1}{2} \sum_{\sigma} \int d^{3} k\left(p_{\mathbf{k} \sigma}^{2}+\omega^{2} q_{\mathbf{k} \sigma}^{2}\right) . \tag{2.7}
\end{equation*}
$$

In this way, we have converted the Hamiltonian function of the classical electromagnetic field into an infinite sum over uncoupled harmonic oscillators with frequencies $\omega=|\mathbf{k}|$ c.
Photon creation and annihilation operators: Since we just found that the electromagnetic field, after Fourier transformation, is equivalent to a set of uncoupled harmonic oscillators, we can use this information to quantize this field by quantizing each of the harmonic oscillators. Hence, we replace the $c$-number functions $q_{\mathbf{k} \sigma}$ and $p_{\mathbf{k} \sigma}$ by operators $\hat{q}_{\mathbf{k} \sigma}$ and $\hat{p}_{\mathbf{k} \sigma}$,

$$
\begin{equation*}
q_{\mathbf{k} \sigma} \mapsto \hat{q}_{\mathbf{k} \sigma}, \quad p_{\mathbf{k} \sigma} \mapsto \hat{p}_{\mathbf{k} \sigma}, \tag{2.8}
\end{equation*}
$$

and postulate their equal-time commutation relations

$$
\begin{equation*}
\left[\hat{q}_{\mathbf{k} \sigma}, \hat{p}_{\mathbf{k}^{\prime} \sigma^{\prime}}\right]:=i \hbar \delta\left(\mathbf{k}-\mathbf{k}^{\prime}\right) \delta_{\sigma, \sigma^{\prime}} . \tag{2.9}
\end{equation*}
$$

Let us now define non-Hermitian operators $\hat{a}_{\sigma}(\mathbf{k})$ and $\hat{a}_{\sigma}^{\dagger}(\mathbf{k})$ in analogy to the complex expansion coefficients $u_{\mathbf{k} \sigma}$

$$
\begin{align*}
& \hat{a}_{\sigma}(\mathbf{k})=\sqrt{\frac{\omega}{2 \hbar}}\left(\hat{q}_{\mathbf{k} \sigma}+\frac{i \hat{p}_{\mathbf{k} \sigma}}{\omega}\right),  \tag{2.10}\\
& \hat{a}_{\sigma}^{\dagger}(\mathbf{k})=\sqrt{\frac{\omega}{2 \hbar}}\left(\hat{q}_{\mathbf{k} \sigma}-\frac{i \hat{p}_{\mathbf{k} \sigma}}{\omega}\right) \tag{2.11}
\end{align*}
$$

With the commutation relations for the operators $\hat{q}_{\mathbf{k} \sigma}$ and $\hat{p}_{\mathbf{k} \sigma}$, we find for the new operators the equal-time commutation relation

$$
\begin{equation*}
\left[\hat{a}_{\sigma}(\mathbf{k}), \hat{a}_{\sigma^{\prime}}^{\dagger}\left(\mathbf{k}^{\prime}\right)\right]=\delta\left(\mathbf{k}-\mathbf{k}^{\prime}\right) \delta_{\sigma, \sigma^{\prime}} \tag{2.12}
\end{equation*}
$$

From the theory of the quantized harmonic oscillator, we know that the operators $\hat{a}_{\sigma}(\mathbf{k})$ and $\hat{a}_{\sigma}^{\dagger}(\mathbf{k})$ destroy and create, respectively, quanta of energy $\hbar \omega=\hbar|\mathbf{k}| c$. That suggests that they can be interpreted as annihilation and creation operators of excitations of the electromagnetic field modes of wavevector $\mathbf{k}$. These elementary field excitations are called photons. We will justify this interpretation in a later lecture.

Meanwhile, we will rewrite both the quantized vector potential and the Hamiltonian in terms of the new variables. For the vector potential in the Schrödinger picture, one finds that

$$
\begin{equation*}
\hat{\mathbf{A}}(\mathbf{r})=\sum_{\sigma} \int \frac{d^{3} k}{(2 \pi)^{3 / 2}} \sqrt{\frac{\hbar}{2 \varepsilon_{0} \omega}} \mathbf{e}_{\sigma}\left[e^{i \mathbf{k} \cdot \mathbf{r}} \hat{a}_{\sigma}(\mathbf{k})+e^{-i \mathbf{k} \cdot \mathbf{r}} \hat{a}_{\sigma}^{\dagger}(\mathbf{k})\right] . \tag{2.13}
\end{equation*}
$$

In principle, we could use any other mode function $\mathbf{A}_{\lambda}(\mathbf{r})$ to write the vector potential as

$$
\begin{equation*}
\hat{\mathbf{A}}(\mathbf{r})=\sum_{\lambda}\left[\mathbf{A}_{\lambda}(\mathbf{r}) \hat{a}_{\lambda}+\mathbf{A}_{\lambda}^{*}(\mathbf{r}) \hat{a}_{\lambda}^{\dagger}\right] . \tag{2.14}
\end{equation*}
$$

If we now write the Hamiltonian as $\hat{H}=\frac{1}{2} \sum_{\lambda}\left(\hat{p}_{\lambda}^{2}+\omega_{\lambda}^{2} \hat{q}_{\lambda}^{2}\right)$ and replace the operators $\hat{q}_{\lambda}$ and $\hat{p}_{\lambda}$ by the new variables $\hat{a}_{\lambda}$ and $\hat{a}_{\lambda}^{\dagger}$,

$$
\begin{align*}
& \hat{q}_{\lambda}=\sqrt{\frac{\hbar}{2 \omega_{\lambda}}}\left[\hat{a}_{\lambda}+\hat{a}_{\lambda}^{\dagger}\right],  \tag{2.15}\\
& \hat{p}_{\lambda}=\frac{1}{i} \sqrt{\frac{\hbar \omega_{\lambda}}{2}}\left[\hat{a}_{\lambda}-\hat{a}_{\lambda}^{\dagger}\right], \tag{2.16}
\end{align*}
$$

we find that

$$
\begin{equation*}
\hat{H}=\frac{1}{2} \sum_{\lambda} \hbar \omega_{\lambda}\left(\hat{a}_{\lambda} \hat{a}_{\lambda}^{\dagger}+\hat{a}_{\lambda}^{\dagger} \hat{a}_{\lambda}\right)=\sum_{\lambda} \hbar \omega_{\lambda}\left(\hat{a}_{\lambda}^{\dagger} \hat{a}_{\lambda}+\frac{1}{2}\right) \tag{2.17}
\end{equation*}
$$

where in the last equation we used the commutation relations (2.12) in the form $\left[\hat{a}_{\lambda}, \hat{a}_{\lambda}^{\dagger}\right]=$ $\delta_{\lambda \lambda^{\prime}}$. The Hamiltonian (2.17) implies the following equations of motion for the photonic creation and annihilation operators

$$
\begin{gather*}
\dot{\hat{a}}_{\lambda}=\frac{1}{i \hbar}\left[\hat{a}_{\lambda}, \hat{H}\right]=-i \omega_{\lambda} \hat{a}_{\lambda} \Rightarrow \hat{a}_{\lambda}(t)=e^{-i \omega_{\lambda} t} \hat{a}_{\lambda},  \tag{2.18}\\
\dot{\hat{a}}_{\lambda}^{\dagger}=\frac{1}{i \hbar}\left[\hat{a}_{\lambda}^{\dagger}, \hat{H}\right]=i \omega_{\lambda} \hat{a}_{\lambda}^{\dagger} \Rightarrow \hat{a}_{\lambda}^{\dagger}(t)=e^{i \omega_{\lambda} t} \hat{a}_{\lambda}^{\dagger} . \tag{2.19}
\end{gather*}
$$

Here we made the distinction between operators in the Heisenberg picture which carry time dependence, and operators in the Schrödinger picture which coincide with their counterparts in the Heisenberg picture at time $t=0$. From the expansion of the vector potential (2.13) we then realize that the photonic amplitude operators appear in combinations $e^{i(\mathbf{k} \cdot \mathbf{r}-\omega t)} \hat{a}_{\sigma}(\mathbf{k})$ and $e^{-i(\mathbf{k} \cdot \mathbf{r}-\omega t)} \hat{a}_{\sigma}^{\dagger}(\mathbf{k})$. For this reason, the annihilation operator $\hat{a}_{\sigma}(\mathbf{k})$ is associated with the positive-frequency components of the electromagnetic field, and the creation operator with negative-frequency components. Heisenberg's equations of motion can be used to write the operator of the electric-field strength as

$$
\begin{equation*}
\hat{\mathbf{E}}(\mathbf{r})=-\dot{\hat{\mathbf{A}}}(\mathbf{r})=i \sum_{\lambda} \omega_{\lambda}\left[\mathbf{A}_{\lambda}(\mathbf{r}) \hat{a}_{\lambda}-\mathbf{A}_{\lambda}^{*}(\mathbf{r}) \hat{a}_{\lambda}^{\dagger}\right] . \tag{2.20}
\end{equation*}
$$

Going back to the Hamiltonian (2.17) one realizes that, because of the commutation relation (2.12), there appears a constant contribution to the total field energy,

$$
\begin{equation*}
\hat{H}_{0}=\frac{1}{2} \sum_{\lambda} \hbar \omega_{\lambda}=\frac{\hbar c}{2} \sum_{\sigma} \int \frac{d^{3} k}{(2 \pi)^{3 / 2}}|\mathbf{k}| . \tag{2.21}
\end{equation*}
$$

Clearly, this contribution is infinitely large. Because it does not even contain any photonic amplitude operators, this energy exist without any photon being present. Hence, this energy relates to the vacuum energy which, one might say, is an artefact of the quantization procedure. Should we be worried? Does this infinite energy have any measurable effect or does it just relocate the point from which we measure relative photon energies? In most cases of interest, this vacuum energy indeed does not play a role if one concentrates on the photons only. We will see later that the presence of material objects or other boundary conditions for the electromagnetic field will make the vacuum energy appear as a measurable quantity. Some additional remarks on field quantization: The quantization procedure outlined above is an example of a canonical quantization in which a system with infinitely many degrees of freedom (a field) is converted into a set of uncoupled harmonic oscillators which are then quantized. The commutation relations postulated for the creation and annihilation operators (2.12) reflect themselves in specific commutation rules between the vector potential and its associated canonical momentum.

The way canonical quantization works in field theories is by starting from a Lagrangian which for electromagnetism in the Coulomb gauge reads

$$
L=\int d^{3} r \mathcal{L}=\frac{1}{2} \int d^{3} r\left\{\varepsilon_{0} \dot{\mathbf{A}}^{2}(\mathbf{r}, \mathbf{t})-\frac{1}{\mu_{0}}[\boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r}, t)]^{2}\right\}
$$

[this is the three-vector version of the relativistic four-vector Lagrangian density $\mathcal{L}=$ $-1 / 4 F^{\mu \nu} F_{\mu \nu}$ with $\left.F^{\mu \nu}=\partial^{\mu} A^{\nu}-\partial^{\nu} A^{\mu}\right]$. The canonical momentum field is derived by functional differentiation (in the space of transverse vector fields) as

$$
\boldsymbol{\Pi}(\mathbf{r}, t)=\frac{\delta L}{\delta \dot{\mathbf{A}}(\mathbf{r}, t)}=\varepsilon_{0} \dot{\mathbf{A}}(\mathbf{r}, t)=-\varepsilon_{0} \mathbf{E}^{\perp}(\mathbf{r}, t)
$$

Then, one postulates the canonical equal-time commutation relations between the vector potential and its momentum field as

$$
\left[\hat{\mathbf{A}}(\mathbf{r}, t), \hat{\boldsymbol{\Pi}}\left(\mathbf{r}^{\prime}, t\right)\right]=i \hbar \boldsymbol{\delta}^{\perp}\left(\mathbf{r}-\mathbf{r}^{\prime}\right)
$$

where $\boldsymbol{\delta}^{\perp}\left(\mathbf{r}-\mathbf{r}^{\prime}\right)$ denotes the transverse $\delta$ function (see mathematical supplement).

## Supplement to Lecture 2: Transverse and longitudinal $\delta$ functions

The correct mathematical definition of the (scalar) $\delta$ distribution is the following: Let $\Phi$ be a test function, i.e. a continuously differentiable function that vanishes with all its derivatives outside a compact set on the real axis $\left[\Phi \in C_{0}^{\infty}(a, b)\right]$. The $\delta$ distribution is a functional over all test function $\Phi$ such that

$$
(\delta, \Phi)=\Phi(0)
$$

Sloppily speaking this means that the $\delta$ 'function' picks out the value of $\Phi$ at the origin. Physicists usually write

$$
\int_{-\infty}^{\infty} d x \delta(x) \Phi(x)=\Phi(0)
$$

which looks similar to the formula above but is mathematically incorrect (it would be correct if $\delta$ were a regular distribution). In any case, the $\delta$ function has a singular point support at the origin. The same is true for the tensor-valued $\delta$ function which we define by

$$
[\boldsymbol{\delta}(\mathbf{r})]_{i j}=\delta_{i j} \delta(\mathbf{r})
$$

Things change when we define the longitudinal tensor-valued $\delta$ function by

$$
\left[\boldsymbol{\delta}^{\|}(\mathbf{r})\right]_{i j} \equiv \delta_{i j}^{\|}(\mathbf{r})=\partial_{i} \partial_{j} \frac{1}{4 \pi|\mathbf{r}|}
$$

Clearly, its support is over the whole real space and not just the origin. Moreover, we see that it is longitudinal as it is actually the gradient of $\boldsymbol{\nabla}(4 \pi|\mathbf{r}|)^{-1}$, and from the vector identity (1.8) we then establish that $\boldsymbol{\nabla} \times \boldsymbol{\delta}^{\|}(\mathbf{r}) \equiv \mathbf{0}$. The transverse tensor-valued $\delta$ function is then defined by the difference between the ordinary and the longitudinal $\delta$ functions,

$$
\delta^{\perp}(\mathbf{r})=\boldsymbol{\delta}(\mathbf{r})-\boldsymbol{\delta}^{\|}(\mathbf{r}) .
$$

It is clearly transverse because we have subtracted the longitudinal part of the full $\delta$ function, hence $\boldsymbol{\nabla} \cdot \boldsymbol{\delta}^{\perp}(\mathbf{r}) \equiv \mathbf{0}$. Transverse and longitudinal $\delta$ function are used to define transverse and longitudinal parts of vector functions $\mathbf{V}(\mathbf{r})$. They are defined as follows:

$$
\begin{aligned}
\mathbf{V}^{\perp}(\mathbf{r}) & =\int d^{3} r^{\prime} \boldsymbol{\delta}^{\perp}\left(\mathbf{r}-\mathbf{r}^{\prime}\right) \mathbf{V}\left(\mathbf{r}^{\prime}\right) \\
\mathbf{V}^{\|}(\mathbf{r}) & =\int d^{3} r^{\prime} \boldsymbol{\delta}^{\|}\left(\mathbf{r}-\mathbf{r}^{\prime}\right) \mathbf{V}\left(\mathbf{r}^{\prime}\right)
\end{aligned}
$$

From the definitions of $\boldsymbol{\delta}^{\perp}(\mathbf{r})$ and $\boldsymbol{\delta}^{\|}(\mathbf{r})$ it is clear that $\mathbf{V}^{\perp}(\mathbf{r})$ and $\mathbf{V}^{\|}(\mathbf{r})$ are indeed transverse and longitudinal vector functions, respectively.

